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Orthogonal Polynomials and Padé Approximants Associated with a System of Arcs*

J. NUTTALL AND S. R. SINGH

Department of Physics, University of Western Ontario, London, Ontario, Canada

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We study the convergence behavior of the diagonal sequence of the Padé table associated with a function with branch points. Given a set of even number, $2l$, of the branch points, a unique set S is constructed which consists of a number of analytic Jordan arcs ending at the branch points. We assume that these arcs are nonintersecting. Let $\sigma(t)$ be a complex, never vanishing function defined on S, satisfying a Lipschitz type smoothness condition there, and let $X(t)$ be the monic polynomial of degree $2l$ with zeros at the branch points. We construct orthogonal polynomials with respect to the weight function $X_+^{-(1/2)}(t)\,\sigma(t)$ and study their asymptotic behavior. The orthogonal polynomials are defined without complex conjugation and the domain of integration is S. Some properties of these polynomials yield convergence in capacity of the diagonal sequence of Padé approximants to $f(t) = \int_S dt' X_+^{-(1/2)}(t')\sigma(t')(t'-t)^{-1}$ in any closed, bounded domain of the complex plain cut along S.

1. INTRODUCTION

The aim of this paper is to gain some understanding of the convergence behavior of diagonal Padé approximants (approximants of the continued fraction) to a function with branch points. We treat functions which can be written in the form

$$f(t) = \int_S dt'\, X_+^{-(1/2)}(t')\, \sigma(t')(t'-t)^{-1}. \tag{1.1}$$

The set S consists of a number of analytic Jordan arcs ending at the given distinct finite points d_i, $i = 1,..., 2l$, in the complex plane. For a given choice of $\{d_i\}$, S is given uniquely by a prescription described in Section 2.

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The weight function σ, defined on S, is constrained to obey smoothness Condition 6.3. This allows, for instance, σ to be an entire function. The polynomial X is

$$X(t) = \prod_{i=1}^{2l} (t - d_i) \tag{1.2}$$

and $X_+^{-(1/2)}$ denotes the value of $X^{-(1/2)}$ on a particular side of S.

The function $f(t)$ has an expansion about the point at ∞

$$f(t) = \sum_{k=1}^{\infty} t^{-k} \int_S dt' X_+^{-(1/2)}(t') \sigma(t') t'^{k-1}. \tag{1.3}$$

From the coefficients in this expansion may be found the $[n/n]$ Frobenius Padé approximant to f.

DEFINITION 1.1. The $[n/n]$ Frobenius Padé approximant to f is

$$[n/n] = V_n(t^{-1})/W_n(t^{-1}) \tag{1.4}$$

where $V_n(t)$, $W_n(t)$ are polynomials in t of degree no higher than n, which satisfy

$$f(t) W_n(t^{-1}) - V_n(t^{-1}) = O(t^{-(2n+1)}) \qquad \text{as } t \to \infty. \tag{1.5}$$

It is known [1] that the Frobenius Padé approximant always exists and is unique, although V_n, W_n may not be unique.

Our main result is Theorem 7.5 where it is shown that $[n/n]$ converges in capacity to f in any closed bounded region of the complex plane not intersecting S. The proof of this theorem is based on the connection between the Padé approximant $[n/n]$ and the orthogonal polynomial $p_n(t)$. The proof given does not apply to the special case in which S does not consist of l nonintersecting analytic arcs.

DEFINITION 1.2. An orthogonal polynomial of order n, $p_n(t)$, is a polynomial of degree $\leqslant n$ satisfying the relations

$$\int_S dt\, X_+^{-(1/2)}(t) \sigma(t) t^k p_n(t) = 0, \qquad k = 0, \ldots, n-1. \tag{1.6}$$

There is always at least one such polynomial not identically zero.

We shall show below that we may choose $W_n(t^{-1}) = t^{-n} p_n(t)$, and that the convergence of the sequence $[n/n]$ follows from a knowledge of the behavior of $p_n(t)$ as $n \to \infty$.

The determination of the asymptotic behavior of $p_n(t)$ forms the bulk of the paper. We use a generalization of the method described by Szegö [2]. We construct an approximation to σ which is the inverse of a polynomial, and.

for large enough n, are able to find the orthogonal polynomials exactly for the approximate weight function. The required p_n is shown to be the solution of an integral equation which can be solved by iteration for large n.

To find orthogonal polynomials for the approximate weight function, we must solve the Jacobi inversion problem for the Riemann surface $\mathscr{R}$ corresponding to $y^2 = X(t)$, which may be thought of as two sheets joined at S. This leads, for each n, to a set of points α_i, $i = 1, ..., l - 1$, on $\mathscr{R}$. At least for an infinite subsequence of integers n, with consecutive members separated by no more than l, the polynomial $p_n(t)$ has, for large n, zeros near to those α_i that lie on the first sheet of $\mathscr{R}$. The other zeros of p_n are near S.

The restriction of the path of integration in (1.1) that joins the points $\{d_i\}$ might appear to be rather severe. However, in the case of weight functions σ with some region of analyticity, this is not the case. For instance, let us consider the case of σ entire. Then it is clear that $p_n(t)$ and $[n/n]$ are unchanged if S is distorted in any way that keeps its ends fixed and does not let S reach ∞. From the Padé approximants or the orthogonal polynomials we cannot tell which choice of integration path was used in their definition, but, at least for the subsequence, the particular set S is chosen as that approached by all but at most $(l - 1)$ of the poles of $[n/n]$ (the zeros of $p_n(t)$) as $n \to \infty$.

This sort of behavior is to be found in the work of Dumas [3], who found explicitly, in terms of elliptic functions, the diagonal Padé approximants to a function involving the square root of a quartic polynomial. The material in Section 5 generalizes this and could be used to obtain his results in more transparent fashion. The same goes for the work of Achyeser [4], which is related to a special case of Dumas' results.

For $l = 1$ the results of this paper have been obtained previously [5], and, for some functions σ satisfying condition (6.3), they are implied by the work of Baxter [6]. For $l > 1$, the results have been derived earlier for a particular choice of σ and a restricted choice of $\{d_i\}$ [7].

In Section 2 we introduce the set S and give some of its properties. In some special cases it has been shown [7] that S may be characterized as the unique set of minimum capacity amongst all sets whose connected components each contain an even number of the points d_i. We expect that this holds in general, but, since this property is not needed for the present work, we have not investigated the question further.

Section 3 gives some properties of orthogonal polynomials (according to our definition) which mirror the properties of the Padé table. These results apply to more general weight functions than those considered here.

The solution of the Jacobi inversion problem and some of its properties needed in the sequel form Section 4. These results allow us to put an upper bound to the length of the intercept of a block of the Padé table with the principal diagonal.

In Section 5, we construct the orthogonal polynomials of sufficiently high order, for the case when $\sigma(t)$ is the inverse of a polynomial, in terms of a function meromorphic on $\mathscr{R}$, including amongst its zeros the points that came from the solution of the Jacobi problem. An explicit form of this function is given, which is needed in the discussion of convergence.

The derivation of the integral equation for p_n in Section 6 is modeled on that of Szegö [2], but the estimates required are more involved. An important result is Lemma 6.7 which permits us to relate the asymptotic behavior of p_n to the Green's function for S with pole at ∞. The method uses a polynomial approximating $\sigma^{-1}(t)$ on S, and the proof of the existence of a suitable polynomial is given in Appendix 2. If we were prepared to strengthen requirement (ii) of Condition 6.3 to apply to the whole of S, this demonstration would be much simpler.

Finally, in Section 7 the convergence proof for Padé approximants is given. This is complicated by the fact that we have information on $p_n(t)$ for a subsequence of the integers n, but the generalization of the recurrence relation between orthogonal polynomials is used to fill in the gaps.

2. The Set S

In this section we define the set S in the complex plane and describe some of its properties. We begin with the Riemann surface $\mathscr{R}$ of y as a function of t, where $y^2 = X(t)$. This consists of two copies of the complex plane. We shall call the points at ∞ on these two sheets ∞_1 and ∞_2. Near ∞_1, we assume $X^{1/2} \sim t^{l}$.

The properties of Abelian (hyperelliptic) differentials and integrals connected with $\mathscr{R}$ will be required in the sequel. An excellent discussion is given by Siegel [8]. A basis for the differentials of the first kind is dw_k, where

$$dw_k = t^{k-1}X^{-1/2}\,dt, \qquad k = 1,\dots, l-1. \tag{2.1}$$

Siegel shows that there exists a unique differential of the third kind dE which is regular apart from simple poles at ∞_1, ∞_2 with residues 1 and -1, and which has all its periods pure imaginary. In terms of dE we define on $\mathscr{R}$ the multivalued hyperelliptic integral of the third kind $\phi(t)$ by

$$\phi(t) = \int_{d_1}^{t} dE. \tag{2.2}$$

Lemma 2.1. *For any path from d_1 to d_i that avoids ∞,*

$$\operatorname{Re}\phi(d_i) = 0, \qquad i = 1,\dots, 2l.$$

Proof. We assume $i \neq 1$, for $\phi(d_1) = 0$. Suppose that $Y(t)$ is a polynomial of degree $l-1$, with coefficient of t^{l-1} unity. Then $YX^{-1/2}\,dt$ is a differential of the third kind which is regular apart from simple poles at ∞_1, ∞_2 with residues 1 and -1. It follows [8] that there exist unique complex constants β_k, $k = 1,..., l-1$, such that

$$YX^{-(1/2)}\,dt - dE = \sum_{k=1}^{l-1} \beta_k\, dw_k\,.$$

Using (2.1) it follows that there exists a unique degree $(l-1)$ polynomial $Z(t)$, coefficient of t^{l-1} unity, such that

$$dE = ZX^{-1/2}\,dt. \tag{2.3}$$

Now a period of dE is defined as the integral of dE round any closed curve on $\mathscr{R}$. A path in the complex plane from d_1 to a point near d_k, a small circle round d_k, and the same path back to d_1 is such a curve. The function $X^{-(1/2)}$ has opposite signs at corresponding points on d_1d_k and d_kd_1. If we let the radius of the small circle approach zero we see that the period corresponding to this path is

$$2\int_{d_1}^{d_i} Z(t)\, X^{-(1/2)}(t)\, dt$$

but we know this is pure imaginary. ∎

LEMMA 2.2. *If* Re $\phi(t) = 0$ *for any given path from* d_1 *to a point in the complex plane, then* Re $\phi(t) = 0$ *for all paths from* d_1 *to* t.

Proof. This follows because the periods of dE are pure imaginary. ∎

DEFINITION 2.3. The set S in the complex plane is

$$S = \{t : \text{Re}\ \phi(t) = 0\}.$$

We have seen that it does not matter which path is used in evaluating $\phi(t)$ in this definition. Our arguments have shown that S is uniquely determined by the points d_i.

Next we give some properties of S. We suppose

$$Z(t) = \prod_{i=1}^{l-1} (t - c_i).$$

LEMMA 2.4. (i) *S is the union of a finite number of finite analytic Jordan arcs, whose endpoints are chosen from the points* d_i, c_i.

(ii) *Each point d_i is the end of one arc and each point c_i in S of multiplicity $(q-1)$ is the end of $2q$ arcs, except that, if $c_i = d_j$, the number of arcs that end at that point is $2q - 1$.*

(iii) *The complement S' of S is connected.*

(iv) Re ϕ *is single-valued in* S'.

Proof. (i) If t_0 is a point where Re $\phi(t_0) = 0$, ϕ is analytic, and $\phi'(t_0) \neq 0$, then the equation $\phi(t) = z$ in a neighborhood of t_0 may be inverted [9] to read

$$t = F(z)$$

with F analytic in a corresponding neighborhood. This shows that the locus Re $z = 0$ is part of an analytic Jordan arc near to t_0 in the t-plane. The only other points on S are d_i and those c_j satisfying Re $\phi(c_j) = 0$, and these points are the only points where the arcs can end.

(ii) Suppose, for a particular value of i, that we have Re $\phi(c_i) = 0$. Since

$$d\phi/dt = ZX^{-(1/2)}$$

we have $d\phi/dt = 0$ at $t = c_i$, and thus near c_i (assuming $c_i \neq d_j$, any j)

$$\phi(t) = \phi(c_i) + A(t - c_i)^q + \cdots, \qquad A \neq 0.$$

This shows that the locus Re $\phi(t) = 0$ consists, in the neighborhood of c_i, of $2q$ analytic Jordan arcs ending at c_i [9]. In a similar way, by expanding ϕ in a power series in $(t - d_i)^{1/2}$, we may prove the rest of the statement.

(iii) Suppose that S', the complement of S, were not connected. Since $\phi \sim \pm \ln t$ as $t \to \infty$, we see that S' extends to ∞ in all directions. Thus S' must contain a bounded domain whose boundary consists of the union of a finite number of Jordan arcs on which Re $\phi = 0$. Since this domain cannot contain a singularity of ϕ as an interior point, we can define in this domain a single-valued analytic function ϕ, whose real part, by Lemma 2.2, is zero on the boundary. This is a contradiction, since Re ϕ is not constant in the domain [10].

(iv) Part (ii) shows that each connected component of S must contain an even number of points d_i. The change in ϕ as we go round a closed contour in S' enclosing one or more components of S is a period of dE and the result follows. ∎

The $2q$ arcs ending at c_i may be regarded as making up q analytic arcs ending elsewhere, and hence the description of S as a number of possibly intersecting arcs with endpoints $\{d_i\}$ is justified.

The function Re $\phi(t)$ is harmonic and single-valued in S', is zero on S and behaves at ∞ like $\ln |t| +$ const. It is therefore [11] the unique Green's function with pole at ∞ for S.

From now on we shall assume that $\mathscr{R}$ is divided into two sheets by S. The function $X^{1/2}$ is single-valued in S'. For later use we arbitrarily assign a direction to each arc of S. We denote by $+$ that side of S which is on the left as we move along the arc, and by $-$ the other side.

If F is a function defined on $\mathscr{R}$, we shall take $F(t_i)$ to mean F evaluated at t on sheet i. Where not specified, $F(t)$ will mean $F(t_1)$. For $t_1 \in S$, $F_+(t_1)$ will mean the limit of $F(t_1)$ as t approaches S from the $+$ side. In the case of $X^{-(1/2)}$, we shall abbreviate $X_+^{-(1/2)}(t_1)$ by $X_+^{-(1/2)}(t)$.

3. Orthogonal Polynomials and Padé Approximants

In Section 7 we shall need some general results, assembled below, on orthogonal polynomials and Padé approximants. The precise form of all the results is not available elsewhere, but their content is not new [1]. Here, we shall make no use of the special character of S or the smoothness of σ. In this section we shall write $\bar{\sigma}$ for $\sigma X_+^{-(1/2)}$.

From any orthogonal polynomial of order n, we can construct the $[n/n]$ Padé approximant to $f(t)$ by the next lemma..

Lemma 3.1. *For any $n \geqslant 0$, the $[n/n]$ Frobenius Padé approximant to $f(t)$ may be written as*

$$[n/n] = p_n^{-1}(t) \int_S dt'\, \bar{\sigma}(t')[p_n(t) - p_n(t')](t' - t)^{-1}. \tag{3.1}$$

Proof. We show that

$$\begin{aligned} W_n(t^{-1}) &= t^{-n} p_n(t), \\ V_n(t^{-1}) &= t^{-n} \int_S dt'\, \bar{\sigma}(t')[p_n(t) - p_n(t')](t' - t)^{-1}, \end{aligned} \tag{3.2}$$

satisfy (1.5). They are both clearly polynomials in t^{-1} of degree no higher than n. We have

$$\begin{aligned} f(t)\, W_n(t^{-1}) - V_n(t^{-1}) &= t^{-n} \int_S dt'\, \bar{\sigma}(t')\, p_n(t')(t' - t)^{-1} \\ &= -t^{-n-1} \int_S dt'\, \bar{\sigma}(t')\, p_n(t') \sum_{k=0}^{\infty} (t'/t)^k \\ &= O(t^{-2n-1}) \end{aligned} \tag{3.3}$$

from (1.6).

The converse of Lemma 3.1 is

Lemma 3.2. *If the $[n/n]$ Frobenius Padé approximant to $f(t)$ can be written in the form $r'(t)/r(t)$, where r, r' are polynomials of degree μ, $\mu - 1$, respectively, $\mu \leqslant n$, then*

$$\int_S dt\, \bar{\sigma}(t)\, r(t)\, t^k = 0, \qquad k = 0, \ldots, n - 1. \tag{3.4}$$

Proof. The uniqueness of the Frobenius Padé approximant [1] shows that we must be able to find $s(t)$, a polynomial in t, of degree $\leqslant n-\mu$ such that the functions appearing in the definition of the Padé approximant may be written

$$W_n(t^{-1}) = s(t^{-1})\, t^{-n} r(t),$$
$$V_n(t^{-1}) = s(t^{-1})\, t^{-n} r'(t),$$

and from (1.5) we obtain

$$s(t^{-1})\, t^{-n} r(t) f(t) - s(t^{-1})\, t^{-n} r'(t) = O(t^{-2n-1}). \tag{3.5}$$

We have for large t

$$\begin{aligned} s(t^{-1})\, t^{-n-1} \int_S dt'\, \bar{\sigma}(t')\, r(t') \sum_{k=0}^{\infty} (t'/t)^k &= s(t^{-1})\, t^{-n} \int_S dt'\, \bar{\sigma}(t')\, r(t')(t-t')^{-1} \\ &= s(t^{-1})\, t^{-n} r(t) \int_S dt'\, \bar{\sigma}(t')(t-t')^{-1} \\ &\quad + s(t^{-1})\, t^{-n} \int_S dt'\, \bar{\sigma}(t')(r(t')-r(t))(t'-t)^{-1} \\ &= s(t^{-1})\, t^{-n} r(t) f(t) - s(t^{-1})\, t^{-n} s'(t) \end{aligned}$$

where $s'(t)$ is a polynomial in t of degree $\mu - 1$. Substituting (3.5) gives

$$t^{-n-1} \sum_{k=0}^{\infty} t^{-k} \int_S dt'\, \bar{\sigma}(t')\, r(t')\, t'^k = t^{-n}(s'(t) - r'(t)) + s^{-1}(t^{-1})\, O(t^{-2n-1})$$

Since $s^{-1}(t^{-1}) = O(t^{n-\mu})$, the result follows on equating powers of t^{-n-k-1}, $k = 0, \ldots, n-1$. ∎

LEMMA 3.3. *For each $n \geqslant 0$ there exists an irreducible orthogonal polynomial of order n, $\bar{p}_n(t)$, unique up to a constant factor. Any orthogonal polynomial $p_n(t)$ of order n may be written as the product of $\bar{p}_n(t)$ and a polynomial in t.*

Proof. Suppose that p_n, r_n are two independent orthogonal polynomials of order n. The uniqueness of the Frobenius Padé approximant [1], with Lemma 3.1, shows that there are polynomials p', r' such that

$$p'(t)/p_n(t) = r_n'(t)/r_n(t). \tag{3.6}$$

Let $\bar{p}_n(t)$ be that factor of $p_n(t)$ remaining after all common factors of p', p_n have been removed. Then (3.6) shows that $\bar{p}_n(t)$ is unique (up to a constant factor). For any $p_n(t)$ we may write

$$p_n(t) = \bar{p}_n(t)\, s(t) \tag{3.7}$$

with $s(t)$ a polynomial.

The fact that $\bar{p}_n(t)$ is an orthogonal polynomial follows from Lemma 3.2. ▮

LEMMA 3.4. (i) *There exists a sequence of nonnegative integers*, ν_i, *called basic integers*, $0 = \nu_1 < \nu_2 < \nu_3 \cdots$, *such that* $\bar{p}_{\nu_i}$ *is of exact degree* ν_i *and* $\bar{p}_n = \bar{p}_{\nu_i}$, $n = \nu_i, \nu_i + 1, \ldots, \nu_{i+1} - 1$. *For no other value of* n *does* $\bar{p}_n = \bar{p}_{\nu_i}$.

(ii) *For integers of the form* ν_i, $\nu_i - 1$, *and no others, the orthogonal polynomial is unique up to a constant factor.*

Proof. (i) Take any n, and suppose that the degree of $\bar{p}_n$ is $\mu \leqslant n$. Also suppose that

$$\int_S dt\, \bar{\sigma}(t)\, t^k \bar{p}_n(t) = 0, \qquad k = 0, 1, \ldots, \lambda - 1 \tag{3.8}$$

with $\lambda \geqslant n$. We assume that (3.8) does not hold for $k = \lambda$.

It follows that $\bar{p}_n(t)$ is an orthogonal polynomial for orders $\mu, \mu + 1, \ldots, \lambda$. Thus it gives rise to $[N/N]$ Padé approximants for $N = \mu, \ldots, \lambda$. All these Padé approximants will have, from Lemma 3.1, the form $\bar{p}'/\bar{p}_n$, with $\bar{p}'$ the same for each N. Since $N = n$ is among the values considered, we know that $\bar{p}'$, $\bar{p}_n$ have no common factors. Thus for each $N = \mu, \ldots, \lambda$, the argument of Lemma 3.3 shows that $\bar{p}_n$ is the irreducible orthogonal polynomial.

(ii) This is obvious for order ν_i. For order $n = \nu_i - 1$, let us write $m = \nu_{i-1}$. Then an orthogonal polynomial for order n is $\bar{p}_m(t)$. Any other independent orthogonal polynomial of order n has the form $\bar{p}_m(t)\, s(t)$, with $s(t)$ a polynomial of exact degree $\geqslant 1$. Suppose that $(t - a)$ is a factor of $s(t)$. Then the argument of Lemma 3.3 shows that there exists a polynomial $r(t)$ such that the $[n/n]$ Padé approximant may be written $r(t)/(\bar{p}_m(t)(t - a))$. Lemma 3.2 shows that

$$\int_S dt\, \bar{\sigma}(t)\, \bar{p}_m(t)(t - a)\, t^k = 0, \qquad k = 1, \ldots, n - 1. \tag{3.9}$$

Since $\bar{p}_m(t)$ is an orthogonal polynomial of order n, we deduce from (3.9) that

$$\int_S dt\, \bar{\sigma}(t)\, \bar{p}_m(t)\, t^k = 0, \qquad k = 1, \ldots, n$$

which contradicts the properties of basic integers discussed above. We conclude that the orthogonal polynomial of order $\nu_i - 1$ is essentially unique.

For integers n other than those of the form ν_i, $\nu_i - 1$, (3.8) shows that $t\bar{p}_{\nu_i}$ is an orthogonal polynomial of degree n, where ν_i is the largest basic integer $< n$. ∎

We now obtain a generalization of the usual recurrence relation satisfied by irreducible orthogonal polynomials.

LEMMA 3.5. *If $\mu < \nu < \lambda$ are consecutive integers from the sequence ν_i of the previous lemma, then a polynomial $Q(t)$ of exact degree $\lambda - \nu$, and a constant C may be found such that*

$$\bar{p}_\lambda = Q\bar{p}_\nu + C\bar{p}_\mu . \tag{3.10}$$

Proof. Consider $\xi = Q\bar{p}_\nu + C\bar{p}_\mu$, where Q, C have the above character. Then for any Q, C the polynomial ξ of degree λ will satisfy

$$\int_S dt\, \bar{\sigma}(t)\, t^k \xi(t) = 0, \qquad k = 0,\ldots, \nu - 2.$$

The conditions

$$\int_S dt\, \bar{\sigma}(t)\, t^k\, \xi(t) = 0, \qquad k = \nu - 1,\ldots, \lambda - 1$$

constitute $\lambda - \nu + 1$ linear relations between the coefficients of Q and C, $(\lambda - \nu + 2)$ unknowns. There is therefore always a nontrivial solution, which is easily seen to be unique. The uniqueness of the orthogonal polynomial of degree λ proves the result. ∎

COROLLARY 3.6. *If $\mu < \nu < \lambda$ are basic integers, not necessarily consecutive, then polynomials Q_μ, Q_λ, and D' of degree $\leqslant \lambda - \nu$, $\nu - \mu$, $\lambda - \mu$, respectively, may be found such that*

$$p_\nu = \frac{Q_\mu p_\mu + Q_\lambda p_\lambda}{D'} . \tag{3.11}$$

D' is not identically zero.

Proof. Suppose that η is the smallest basic integer $> \nu$. Repeated use of (3.10) shows that

$$p_\lambda = Q^{(1)} p_\eta + Q^{(2)} p_\nu , \tag{3.12}$$

where $Q^{(1)}$ is a polynomial of exact degree $\lambda - \eta$, and $Q^{(2)}$ is a polynomial of degree $\leqslant \lambda - \eta - 1$. Similarly, we have

$$Cp_\mu = Q^{(3)} p_\eta + Q^{(4)} p_\nu \tag{3.13}$$

where $Q^{(3)}$ is a polynomial of degree $\leqslant \nu - \mu - 1$, $Q^{(4)}$ a polynomial of exact degree $\nu - \mu$, and the constant C may be zero.

Solving (3.12) and (3.13) gives

$$p_\nu = \frac{Q^{(3)}p_\lambda - CQ^{(1)}p_\mu}{Q^{(3)}Q^{(2)} - Q^{(1)}Q^{(4)}} .$$

We deduce that $D' = Q^{(3)}Q^{(2)} - Q^{(1)}Q^{(4)}$ is not identically zero from the fact that, if it were, we would have

$$Q^{(4)}p_\lambda = CQ^{(2)}p_\mu .$$

The equation cannot hold, because the degree of the left side is greater than the degree of the right. ▌

4. The Jacobi Inversion Problem

In this section we discuss the solution of an example of the Jacobi inversion problem. We shall need this solution and some of its properties when we construct orthogonal polynomials.

Suppose that S has p connected components and that $d_1 ,..., d_p$ each belong to a different component. (We assume that only one arc of S ends at d_1.) Let $C = (C_1 ,..., C_{p-1})$ be a set of finite arcs joining d_1d_2 , $d_2d_3 \cdots$, $d_{p-1}\, d_p$, that do not intersect each other or S (except at their ends). We shall need hyperelliptic integrals of the first kind, $u_k(\alpha)$, defined uniquely for a point α on either sheet of $\mathscr{R}$ by

$$u_k(\alpha) = \int_{\infty_2}^{\alpha} dw_k . \tag{4.1}$$

If α is on the second sheet, the path of integration in (4.1) runs from ∞_2 to α without crossing S or the arcs of C. If α is on the first sheet, the path of integration does not cross the arcs C, but crosses S once near the point d_1 .

We also define $2(l - 1)$ arcs in the complex plane, L_j , $j = 1,..., 2(l - 1)$, so that L_j joins the points d_1 , d_{j+1} and does not intersect S except at its ends. A set of periods of the integrals of the first kind is given by

$$\Omega_{kj} = 2 \int_{L_j} dw_k \tag{4.2}$$

where $X^{-(1/2)}$ in dw_k is evaluated on the first sheet.

Central to our discussion is the solution of the Jacobi inversion problem of finding on $\mathscr{R}$ points α_i , $i = 1,..., l - 1$, along with integers η_j , $j = 1,...,$ $2(l - 1)$, satisfying

$$\sum_{i=1}^{l-1} u_k(\alpha_i) = W_k{}^m + \sum_{j=1}^{2(l-1)} \eta_j \Omega_{kj} + nu_k(\infty_1), \qquad k = 1,..., l - 1, \tag{4.3}$$

for different values of integers m, n. W_k^m, for each k, is a sequence of complex numbers such that

$$W_k^m \underset{m \to \infty}{\longrightarrow} W_k . \tag{4.4}$$

It follows from the Jacobi inversion theorem [8] that (4.3) may always be solved, perhaps not uniquely.

An integral divisor is defined to be a set of points $\beta = \beta_1 \cdots \beta_{l-1}$ on $\mathscr{R}$. Let us denote by $U(\beta)$ the vector in $C^{(l-1)}$ with components $\sum_{i=1}^{l-1} u_k(\beta_i)$ and $x = x_1 \cdots x_{l-1}$. If x_i, $i = 1,\ldots, l-1$ vary independently over $\mathscr{R}$, $U(x)$ varies over a domain J in $C^{(l-1)}$. Let Ω_j denote the vector in $C^{(l-1)}$ with components Ω_{kj}. From the theory of Abelian integrals [8] we have that Ω_j, $j = 1,\ldots, 2(l-1)$ are linearly independent over the field of reals, and that any vector in $C^{(l-1)}$ can be written uniquely as the sum of a vector in J and an integral linear combination of Ω_j. Two vectors v_1, v_2 in $C^{(l-1)}$ which correspond in this way to the same vector in J are said to be congruent, and we shall write $v_1 \equiv v_2$. Also in $C^{(l-1)}$ we shall use the norm $\|\cdot\|$ defined by $\|v\| = \{\sum_{k=1}^{l-1} |v_k|^2\}^{1/2}$. We are now in a position to prove

LEMMA 4.1. *For each solution of* (4.3) *there exist real numbers* b_j, $\xi_j^{m,n}$ *such that*

$$\eta_j = -nb_j + \xi_j^{m,n}, \qquad j = 1,\ldots, 2(l-1). \tag{4.5}$$

and for large enough m

$$|\xi_j^{m,n}| \leq \lambda \|W^m\| . \tag{4.6}$$

b_j, λ *are constants independent of* n *and* m.

Proof. Equation (4.3) may be written as

$$U(x) = W^m + \sum_{j=1}^{2(l-1)} \eta_j \Omega_j + nU(\infty_1). \tag{4.7}$$

Since Ω_j, $j = 1,\ldots, 2(l-1)$, are linearly independent on the field of reals, and $U(\infty_1)$, $(U(x) - W^m)$ are vectors in $C^{(l-1)}$, there exist unique real numbers b_j, $\xi_j^{m,n}$, $j = 1,\ldots, 2(l-1)$ such that

$$U(\infty_1) = \sum_{j=1}^{2(l-1)} b_j \Omega_j \tag{4.8}$$

and

$$U(x) - W^m = \sum_{j=1}^{2(l-1)} \xi_j^{m,n} \Omega_j . \tag{4.9}$$

Substituting in (4.7) and equating the coefficients of Ω_j for each j, leads to the relation (4.5).

Also since $U(\alpha) \in J$, $\xi_j^{m,n}$ can be bounded by a constant independent of n. The proof is completed by observing, further, that $W^m \to_{m\to\infty} W$. ▌

If (4.3) has two different solutions, say α, β, then we have

$$U(\alpha) = U(\beta). \tag{4.10}$$

Abel's theorem [8] shows that (4.10) can hold if and only if the two integral divisors $\alpha = (\alpha_1 \cdots \alpha_{l-1})$ and $\beta = (\beta_1 \cdots \beta_{l-1})$ are equivalent, i.e., $\alpha \sim \beta$. The following lemma determines the equivalence class of an integral divisor in the case under consideration, the hyperelliptic $\mathscr{R}$. It uses the sheet interchange operator h defined by [12].

DEFINITION 4.2. We say that $\beta = h(\alpha)$ if α and β correspond to the same point in the complex plane and $X^{1/2}$ has opposite signs.

LEMMA 4.3. (i) *Suppose the divisor* $\alpha = (\alpha_1 \cdots \alpha_{l-1})$ *may be written as*

$$\alpha = \gamma_1 h(\gamma_1)\, \gamma_2 h(\gamma_2) \cdots \gamma_j h(\gamma_j)\, \alpha_{2j+1} \cdots \alpha_{l-1}, \qquad j \leqslant \tfrac{1}{2}(l-2)$$

where

$$\alpha_i \neq h(\alpha_k),\ i \neq k, \qquad i, k = 2j+1, \ldots, l-1.$$

Then a divisor β *is equivalent to* α *if and only if it has the form*

$$\beta = \beta_1 h(\beta_1) \cdots \beta_j h(\beta_j)\, \alpha_{2j+1} \cdots \alpha_{l-1}.$$

(ii) *If* $\alpha \sim \beta$ *and* α *has no paired points* (*see Appendix* 1) *then* $\alpha = \beta$.

We feel that this result should be known, but, since we do not know where a proof is to be found, one is given in Appendix 1.

To proceed, let us define subsets S_{jk} of $C^{(l-1)}$ as follows.

DEFINITION 4.4. For $j \leqslant l-1$, $k \leqslant l-j-1$,

$$S_{jk} = \{s : s = U((\infty_2)^j (\infty_1)^k \alpha_{j+k+1} \cdots \alpha_{l-1}),\ \alpha_{j+k+1}, \ldots, \alpha_{l-1} \in \mathscr{R}\}.$$

LEMMA 4.5. (i) *For* $j \geqslant 1, k \geqslant 0$ *we have that if* $s \in S_{j-1,k+1}$ *then* $s - U(\infty_1) \in S_{jk}$, *and, if* $s \in S_{jk}$, *then* $s + U(\infty_1) \in S_{j-1,k+1}$.

(ii) *If* $s \in S_{0k}$ *and* $s \in S_{10}$, *then* $s \in S_{1k}$, $k < l-1$.

Proof. (i) If $s \in S_{j-1,k+1}$ then there must be $\alpha_{j+k+1}, \ldots, \alpha_{l-1}$ such that (note that $U(\infty_2) = 0$)

$$s = kU(\infty_1) + \sum_{i=j+k+1}^{l-1} U(\alpha_i),$$

i.e.,

$$s - U(\infty_1) = (k-1)\, U(\infty_1) + \sum_{i=j+k+1}^{l-1} U(\alpha_i)$$

and the first part of (i) follows ($U(\infty_2) = 0$). The second part of (i) is proved in the same way.

(ii) There must be a divisor $\alpha = (\infty_1)^k \alpha_{k+1} \cdots \alpha_{l-1}$ and another divisor $\beta = \infty_2\beta_2 \cdots \beta_{l-1}$ such that

$$s = U(\alpha) = U(\beta).$$

Thus $\alpha \sim \beta$, and we conclude that either $\alpha = \beta$ or from Lemma 4.3(ii), α contains at least two points that are images of one another under h. In the first case the result follows, and in the second we must have either $\alpha_{k+1} = h(\infty_1)$ or $\alpha_{k+1} = h(\alpha_{k+2})$. The first of these alternatives gives $\alpha_{k+1} = \infty_2$ and the second means that $\alpha \sim (\infty_1)^{k+1}\, \infty_2\, \alpha_{k+3} \cdots \alpha_{l-1}$, giving the result each time. ∎

With this we are able to prove

LEMMA 4.6. *Suppose that we have a sequence of vectors $s_j \in C^{(l-1)}$, $j = 0,\ldots, l-1$, such that*

$$s_{j+1} = s_j + U(\infty_1), \qquad j = 0,\ldots, l-2. \tag{4.11}$$

Then there is at least one j such that $s_j \notin S_{10}$.

Proof. Assume the contrary that $s_j \in S_{10}$, $j = 0,\ldots, l-1$. Since $s_0 \in S_{10}$, Lemma 4.5(i) gives that $s_1 \in S_{01}$. Since we also are assuming that $s_1 \in S_{10}$, we deduce from Lemma 4.5(ii) that $s_1 \in S_{11}$. Equation (4.11) then tells us that $s_2 \in S_{02}$ and from the assumption we find $s_2 \in S_{12}$. In this manner we deduce that $s_{l-1} \in S_{0,l-1}$ and $s_{l-1} \in S_{10}$, implying the existence of a divisor that is equivalent to but different from $(\infty_1)^{l-1}$, which is impossible (Lemma 4.3(ii)). ∎

Let us define for all sufficiently small $\epsilon > 0$, a set $N_1^\epsilon \subset \mathscr{R}$ near to ∞_1 by

$$N_1^\epsilon = \{\alpha : \alpha \text{ is on the first sheet of } \mathscr{R} \text{ and } |\alpha| < \epsilon^{-1}\}.$$

In the same way we define N_2^ϵ near ∞_2. We introduce a set $S_{jk}^\epsilon \subset C^{(l-1)}$ as follows.

DEFINITION 4.7. $S_{jk}^\epsilon = \{s : s = U(\alpha),\ \alpha = \alpha_1 \cdots \alpha_{l-1};\ \alpha_i \in N_2^\epsilon,\ i = 1,\ldots, j;\ \alpha_i \in N_1^\epsilon,\ i = j+1,\ldots, j+k\}$.

LEMMA 4.8. *There exists a $\delta_{jk}(\epsilon)$ such that, if $s \in S_{jk}^\epsilon$ then* $\operatorname{dist}(s, S_{jk}) < \delta_{jk}(\epsilon)$, *and* $\delta_{jk}(\epsilon) \underset{\epsilon\to 0}{\longrightarrow} 0$.

Proof. There is an $s' \in S_{jk}$ such that $s' \equiv U((\infty_2)^j (\infty_1)^k \alpha_{j+k+1} \cdots \alpha_{l-1})$ where the α_i are those that appear in Definition 4.7 and

$$s - s' = \sum_{i=1}^{j} (U(\alpha_i) - U(\infty_2)) + \sum_{i=j+1}^{k} (U(\alpha_i) - U(\infty_1))$$

and the result follows from the continuity of $U(\alpha_i)$ on $\mathscr{R}$. ▌

By symmetry, we have $\delta_{10}(\epsilon) = \delta_{01}(\epsilon)$.

Now we are ready to consider the solution of (4.6), which we denote by $\alpha^{m,n}$. Remember that this may not be unique. The result at which most of this section has been aimed is

LEMMA 4.9. (i) *It is possible to find m_0 and $\epsilon > 0$ such that for all $m > m_0$, there is at least one value of k from amongst $m, \ldots, m + l - 1$, for which all solutions of* (4.6) *satisfy the condition*

$$\alpha_i^{k,k} \notin N_2^{\epsilon} \quad \text{and} \quad \alpha_i^{k,k+1} \notin N_1^{\epsilon}, \quad i = 1, \ldots, l-1.$$

(ii) *The solutions $\alpha^{k,k}$, $\alpha^{k,k+1}$ of* (i) *are unique for each k found.*

Proof. (i) We suppose that the result is not true. This means that there is an infinite sequence of values of m (depending on ϵ) for which, no matter how small ϵ, for all $k = m, \ldots, m + l - 1$, there is at least one solution satisfying at least one of $\alpha_1^{k,k} \in N_2^{\epsilon}$, $\alpha_1^{k,k+1} \in N_1^{\epsilon}$. This means that at least one of $U(\alpha^{k,k}) \in S_{10}^{\epsilon}$, $U(\alpha^{k,k+1}) \in S_{01}^{\epsilon}$ holds. We can therefore find an infinite sequence of values of m and a function $\epsilon(m) \to 0$ as $m \to \infty$, such that at least one of

$$U(\alpha^{k,k}) \in S_{10}^{\epsilon(m)}, \qquad U(\alpha^{k,k+1}) \in S_{01}^{\epsilon(m)}, \qquad k = m, \ldots, m + l - 1. \tag{4.12}$$

Since $U(\alpha) \in J$, there must be a subsequence of this sequence, which we still denote by m, and a fixed vector $s_0 \in C^{(l-1)}$ such that either $| U(\alpha^{m,m}) - s_0 | \to 0$ or $| U(\alpha^{m,m+1}) - s_0 | \to 0$ as $m \to \infty$. Suppose that the first alternative holds. The argument is similar in the second case.

From (4.7) we have

$$U(\alpha^{m+i,m+j}) - U(\alpha^{m,m}) \equiv W^{m+i} - W^m + jU(\infty_1).$$

Thus for each $j = 0, \ldots, l - 1$, $U(\alpha^{m+i,m+j})$ converges sequentially to s_j where $s_j \equiv s_0 + jU(\infty_1)$. Now since $s_{10}^{\epsilon(m)}, s_{01}^{\epsilon(m)} \to_{m\to\infty} s_{10}, s_{01}$, from (4.12) we have that

$$s_j \in S_{10} \quad \text{or} \quad s_{j+1} \in S_{01}, \qquad j = 0, \ldots, l-1. \tag{4.13}$$

Since $s_{j+1} \equiv s_j + U(\infty_1)$, Lemma 4.5 shows that one alternative of (4.13) implies the other, so that both hold. But this is a contradiction (Lemma 4.6).

(ii) If the divisor $\alpha^{k,k}$ of (i) is not unique it follows from Lemma 4.3 that $\alpha^{k,k}$ is in an equivalence class of divisors with at least one pair $\alpha_1 h(\alpha_1)$. Setting $\alpha_1 = \infty_1$ we have that $\alpha_1^{k,k} \in N_2^0$ which contradicts the result of (i). Similar arguments prove the uniqueness of $\alpha^{k,k-1}$. ∎

5. Orthogonal Polynomials for Special Weights

In this section we construct explicitly (at least in terms of the solution of the Jacobi inversion problem) orthogonal polynomials for the case in which $\sigma^{-1}(t) = \rho_m(t)$, where $\rho_m(t)$ is a polynomial of degree $\bar{m} = m - l + 1$ that does not vanish on S. From this we could, using Christoffel's formula [13], deal with the case in which σ was a rational fraction with no poles on S, but we do not need this for the sequel. This work is a generalization of that of Szegö [14] for the unit circle and Dumas [3] for the case $l = 2$.

Let us suppose that the zeros of ρ_m are at r_i, $i = 1,..., \bar{m}$. Then the Jacobi inversion theorem tells us that there is at least one divisor $\alpha = \alpha_1 \cdots \alpha_{l-1}$, $\alpha_i \in \mathscr{R}$, satisfying, for $n \geq m$

$$\alpha_1 \cdots \alpha_{l-1} r_1 \cdots r_{\bar{m}} (\infty_2)^{n-m} \sim (\infty_1)^n. \tag{5.1}$$

In the divisor on the left, we assume that $r_1 ,..., r_{\bar{m}}$ are all on the second sheet of $\mathscr{R}$.

It follows [8] that there is a function $F(t)$, meromorphic on $\mathscr{R}$, unique up to a constant factor, that has zeros at the points of the divisor on the left and poles at the divisor on the right. The function $R(h(t))$ has the same zeros and poles with the sheets reversed [12]. The function $F(t) F(h(t))$ therefore has zeros at α_i, $h(\alpha_i)$, $i = 1,..., l-1$ and r_i, $h(r_i)$, $i = 1,..., \bar{m}$ and poles of order m at ∞_1, ∞_2. Thus

$$F(t) F(h(t)) = P(t) \rho_m(t) \text{ const} \tag{5.2}$$

where

$$P(t) = \prod_{i=1}^{l-1} (t - \alpha_i). \tag{5.3}$$

Theorem 5.1. *The function $q_n(t) = F(t_1) + F(h(t_1))$ is an orthogonal polynomial of order n for the weight $\sigma_m = \rho_m^{-1}$, $n \geq m$, where ρ_m is a polynomial of degree $m - l + 1$.*

Proof. The function $q_n(t)$ is defined in the complex plane cut at S. As t approaches S from the + side $F(t_1) \to F_+(t_1)$ and $F(h(t_1)) \to F_-(t_1)$, $t_1 \in S$. Thus, on S,

$$q_n(t) = F_+(t_1) + F_-(t_1) \tag{5.4}$$

and q_n has no discontinuity across S. Since q_n is analytic everywhere except at ∞, where $q_n = O(t^n)$, we deduce that q_n is a polynomial of degree no higher than n.

To check orthogonality, we compute

$$I = \int_S dt\, X_+^{-(1/2)}(t)\, \sigma_m(t)\, t^k q_n(t).$$

We use (5.2) to obtain on S,

$$\begin{aligned}\sigma(t)\, q_n(t) &= \rho_m^{-1}(t)(F_+(t_1) + F_-(t_1))\\ &= P(t)(F_-^{-1}(t_1) + F_+^{-1}(t_1)).\end{aligned}$$

Thus we have

$$I = -\tfrac{1}{2} \int_\Gamma dt\, X^{-(1/2)}(t_1)\, P(t)\, F^{-1}(t_1)\, t^k \tag{5.5}$$

where Γ is a contour enclosing S, which may be distorted to the circle at infinity, since $P(t)\, F^{-1}(t_1)$ is analytic outside S. If this is done we see that $I = 0$, $k < n$, as required, since $P(t)\, F^{-1}(t_1) \sim t^{-n+l-1}$ as $t \to \infty$. ∎

We remark that q_n is of exact degree n unless at least one $\alpha_i = \infty_1$.

In the next section, we need to consider a sequence of ρ_m, and in order to discuss convergence, we will require a more explicit formula for F. We assume $\alpha_1, \ldots, \alpha_\nu$ are on the first sheet.

LEMMA 5.2. *The function $F(t)$ may be written on the first sheet as*

$$F(t) = \exp(X^{1/2}(t_1)\, \psi(t)) \prod_{i=1}^{l-1} \theta(\alpha_i) \tag{5.6}$$

where

$$\begin{aligned}\psi(t) = {} & -\frac{1}{2\pi i} \int_S dt'\, (t' - t)^{-1}\, X_+^{-(1/2)}(t') \Big\{ \ln \sigma_m(t') \\ & + \sum_{i=1}^{\nu} \ln(t' - \alpha_i) - \sum_{i=\nu+1}^{l-1} \ln(t' - \alpha_i) + 2(n - \nu) \ln(t' - d_1) \Big\} \\ & + X^{-(1/2)}(t_1) \Big\{ \sum_{i=1}^{\nu} \ln(t - \alpha_i) + (n - \nu) \ln(t - d_1) \Big\} \\ & - \sum_{j=1}^{2(l-1)} \eta_j \int_{L_j} dt'\, (t' - t)^{-1}\, X^{-(1/2)}(t_1').\end{aligned} \tag{5.7}$$

Here, α_i, $i = 1, \ldots, l - 1$, and η_j, $j = 1, \ldots, 2(l - 1)$ are chosen to satisfy (4.3) with

$$W_k^m = -(1/i\pi) \int_S dt\, X_+^{-(1/2)}(t)\, t^k \ln \sigma_m(t). \tag{5.8}$$

We have used the continuous function $\theta(\alpha)$ which satisfies $\theta(\alpha) = \alpha^{-(1/2)}$ if $|\alpha| > 1$.

In the integral, $\ln(t' - \alpha_i)$ is to be made single-valued by placing a cut from α_i to ∞ that does not cross S or C.

If any α_i should be at infinity, the appropriate limit must be taken. Similarly for $\alpha_i \in S$ but not at an endpoint, the limit as $\alpha_i \to S$ from a side determined by the location of α_i on $\mathscr{R}$ must be taken.

Proof. From its form $F(t)$ of (5.6) is analytic and single-valued in S' cut along the arcs L_j. Plemelj's formula [15] shows that F does not change as we cross L_j, and so F is analytic and single-valued in S'. If $t \in S$ we find, using Plemelj's formula, that

$$F_+(t)\, F_-(t) = \prod_{i=1}^{l-1} \theta^2(\alpha_i)\, P(t)\, \sigma_m^{-1}(t). \tag{5.9}$$

For large t the integrals in (5.7) have an expansion

$$\sum_{k=1}^{\infty} a_k t^{-k}$$

where

$$\begin{aligned} a_k = (1/2\pi i) \int_S dt'\, X_+^{-(1/2)}(t')\, t'^{k-1} \Big\{ \ln \sigma_m(t') &+ \sum_{i=1}^{\nu} \ln(t' - \alpha_i) \\ &- \sum_{i=\nu+1}^{l-1} \ln(t' - \alpha_i) + 2(n - \nu) \ln(t' - d_1) \Big\} \\ &+ \sum_{j=1}^{2(l-1)} \eta_j \int_{L_j} dt'\, X^{-(1/2)}(t_1')\, t'^{k-1}. \end{aligned} \tag{5.10}$$

If α is a point not on S, we have

$$\int_S dt'\, X_+^{-(1/2)}(t')\, t'^{k-1} \ln(t' - \alpha) = -\tfrac{1}{2} \int_\Gamma dt'\, X^{-(1/2)}(t_1')\, t'^{k-1} \ln(t' - \alpha),$$

where Γ is a contour surrounding S but not including α. By distorting Γ to ∞ except for sections from ∞ to α and α to ∞, we find, for $1 \leqslant k \leqslant l$,

$$\int_S dt'\, X_+^{-(1/2)}(t')\, t'^{k-1} \ln(t' - \alpha) = -\pi i \int_\infty^\alpha dt'\, X^{-(1/2)}(t_1')\, t'^{k-1}.$$

If $\alpha \in S$, the same result holds on taking a limit.

Using this formula, we obtain

$$\begin{aligned} a_k = (1/2\pi i) \int_S dt\, X_+^{-(1/2)}(t)\, t^{k-1} \ln \sigma_m(t) \\ - \tfrac{1}{2} \sum_{i=1}^{\nu} \Big\{ \int_\infty^{\alpha_i} dt\, X^{-(1/2)}(t_1)\, t^{k-1} - 2 \int_\infty^{d_1} dt\, X^{-(1/2)}(t_1)\, t^{k-1} \Big\} \\ + \tfrac{1}{2} \sum_{i=\nu+1}^{l-1} \int_\infty^{\alpha_i} dt\, X^{-(1/2)}(t_1)\, t^{k-1} - n \int_\infty^{d_1} dt\, X^{-(1/2)}(t_1)\, t^{k-1} + \tfrac{1}{2} \sum_{j=1}^{2(l-1)} \eta_j \Omega_{kj}. \end{aligned}$$

Now $X^{-(1/2)}(t_1) = -X^{-(1/2)}(t_2)$, so that

$$\int_{\infty}^{\alpha_i} dt\, X^{-(1/2)}(t_1)\, t^{k-1} - 2\int_{\infty}^{d_1} dt\, X^{-(1/2)}(t_1)\, t^{k-1}$$
$$= \int_{\infty_2}^{d_1} dt\, X^{-(1/2)}(t)\, t^{k-1} + \int_{d_1}^{\infty_1} dt\, X^{-(1/2)}(t)\, t^{k-1} + \int_{\infty_1}^{\alpha_i} dt\, X^{-(1/2)}(t)\, t^{k-1}$$
$$= u_k(\alpha_i)$$

where α_i is to be placed on the first sheet of $\mathscr{R}$. Thus, with α_i assigned to the sheets of $\mathscr{R}$ as above,

$$a_k = (1/2\pi i)\int_S dt\, X_+^{-(1/2)}(t)\, t^{k-1} \ln \sigma_m(t) - \tfrac{1}{2}\sum_{i=1}^{l-1} u_k(\alpha_i)$$
$$+ n\int_{\infty_2}^{d_1} dw_k + \tfrac{1}{2}\sum_{j=1}^{2(l-1)} \eta_j \Omega_{kj}, \qquad k < l$$

and this is zero from (4.3).

We deduce that, as $t \to \infty$, the contribution to F from the integral terms in (5.7) is bounded. The remaining factor in F is

$$(t - d_1)^{n-\nu} \prod_{i=1}^{\nu} (t - \alpha_i) \text{ const}$$

so that $F(t) = O(t^n)$ as $t \to \infty$.

Thus the function $F(t)$ has zeros at $\alpha_1, \ldots, \alpha_\nu$ and an nth order pole at ∞. We define what we shall show to be its continuation to the second sheet of $\mathscr{R}$ by

$$H(t) = \prod_{i=1}^{l-1} \theta^2(\alpha_i)\, P(t)\, \rho_m(t)\, F^{-1}(t). \tag{5.11}$$

This function is analytic and single-valued in S', with zeros at $r_1, \ldots, r_{\bar{m}}$ and $\alpha_{\nu+1}, \ldots, \alpha_{l-\nu}$, and a zero of order $(n - m)$ at ∞. Equation (5.9) shows that $F(h(t)) = H(t)$ provides the required continuation. On $\mathscr{R}$, $F(t)$ of (5.6) is meromorphic with poles and zeros as required and is therefore the function introduced at the beginning of the section.

The argument is easily modified if an α_i should be on S or at ∞. ∎

6. Orthogonal Polynomials for General Weights

In this section we come to the problem of finding orthogonal polynomials $p_n(t)$ for a weight function $\sigma(t)$ satisfying Condition 6.3 below. Our procedure follows that of Szegö [2] for the case of the unit circle. An integral equation satisfied by $p_n(t)$ is derived. The complication arises that, even for large n, the integral equation may not always be solved by iteration. However, we show that solution is possible for an infinite sequence of integers m, with consecutive integers differing by no more than a constant, and information about the remaining polynomials is derived in Section 7 using the results of Section 3.

We shall use the polynomials of the previous section for a sequence of weight functions $\sigma_m(t)$, specified in Condition 6.3, that approximate $\sigma(t)$. The polynomials of degree m, $m-1$, associated with weight $\sigma_m(t)$ we shall call $q(t)$, $q'(t)$. The corresponding functions F, P will be denoted by F, F' and P, P'. Similarly we shall use x_i, x_i' for $x_i^{m,m}$, $x_i^{m,m-1}$.

With W_δ^m defined from the weight $\sigma_m(t)$ of Condition 6.3, we denote by Σ the sequence of integers given by Lemma 4.9.

LEMMA 6.1. *For $m \in \Sigma$, provided m is large enough, we have*

$$\mu_m p_m(t) = q(t) + \int_S dt'\, X_+^{(1/2)}(t')(\sigma(t') - \sigma_m(t'))\, K(t,t')(t'-t)^{-1} p_m(t') \qquad (6.1)$$

where

$$K(t,t') = (q(t)\, q'(t') - q'(t)\, q(t')) \qquad (6.2)$$

and μ_m is a constant.

Proof. The function $K(t,t')(t'-t)^{-1}$ is a polynomial in t' of degree m, and the coefficient of t'^m is $\bar{q}q(t)$, where $\bar{q} \neq 0$ is the coefficient of t^{m-1} in $q'(t)$. From the orthogonality of p_m, we find

$$\begin{aligned} \int_S dt'\, X_+^{(1/2)}(t')\, \sigma(t')\, K(t,t')(t'-t)^{-1} p_m(t') \\ = \bar{q}q(t) \int_S dt'\, X_+^{(1/2)}(t')\, \sigma(t')\, t'^m p_m(t'). \end{aligned} \qquad (6.3)$$

We shall show later that the integral on the right-hand side of (6.3) cannot be zero for large enough m. We therefore normalize p_m so that the r.h.s. of (6.3) is $-q(t)$.

For $t \in S$, we have

$$\begin{aligned} &\int_S dt'\, X_+^{(1/2)}(t')\, \sigma_m(t')\, K(t,t')(t'-t)^{-1} p_m(t') \\ &\quad = -\tfrac{1}{2} \int_\Gamma dt'\, X^{(1/2)}(t_1')(q(t)\, P'(t')\, F^{-1}(t') - q'(t)\, P(t')\, F'^{-1}(t')) \\ &\quad \times (t'-t)^{-1} p_m(t') \end{aligned}$$

where Γ is a closed contour including S but not t. We have used an argument similar to that of Theorem 5.1. No contribution results if Γ is distorted to ∞, but to compensate we must add the residue at $t' = t$. This gives (6.1) with

$$\mu_m = i\pi X^{(1/2)}(t_1)(P(t)\, F^{-1}(t)\, q'(t) - P'(t)\, F'^{-1}(t)\, q(t)).$$

Using (5.2) and the definition of q, we obtain

$$\mu_m = i\pi X^{-(1/2)}(t_1)\,\rho_m^{-1}(t)(F(h(t))(F'(t) + F'(h(t))) - F'(h(t))(F(t) + F(h(t)))) = i\pi X^{-(1/2)}(t_1)\,\rho_m^{-1}(t)(F(h(t))\,F'(t) - F'(h(t))\,F(t)). \quad (6.4)$$

This is a function meromorphic on $\mathscr{R}$, invariant under $t \to h(t)$, and bounded at ∞. The only such function is a constant. ∎

Before proceeding, we shall obtain some information on μ_m, the constant of (6.4).

LEMMA 6.2. *For $m \in \Sigma$, we can find $\mu_0 > 0$ such that $|\mu_m| > \mu_0$, if $m > m_0$.*

Proof. We begin with the formula (6.4) and use (5.9) to obtain

$$\mu_m = i\pi X^{-(1/2)}(t_1)(\Omega P(t)\,F'(t)\,F^{-1}(t) - \Omega' P'(t)\,F(t)\,F'^{-1}(t))$$

where

$$\Omega = \prod_{i=1}^{l-1} \theta^2(\alpha_i).$$

We shall evaluate μ_m by taking $t \to \infty_1$. The second term vanishes, and the first gives

$$\mu_m = i\pi \exp(a_l' - a_l) \prod_{i=1}^{l} \theta(\alpha_i)\,\theta(\alpha_i')$$

where a_l, a_l' are given by (5.10) with $n = m, m + 1$.

Because of the convergence of σ_m to σ, we see that for large m, $a_l' - a_l$ can be large only if some α_i, α_i' are large, since the explicit n dependence cancels, and $\eta_j' - \eta_j$ cannot be large from (4.5).

Thus we consider the form for large α of

$$(1/2\pi i) \int_S dt'\, X_{\mp}^{-(1/2)}(t')\, t'^{l-1} \ln(t' - \alpha) \sim (\ln \alpha)(2\pi i)^{-1} \int_S dt'\, X_{+}^{-(1/2)}(t')\, t'^{l-1} + O(\alpha^{-1}) \sim -\tfrac{1}{2} \ln \alpha + O(\alpha^{-1}).$$

So from (5.6) we deduce

$$\mu_m = \prod_{i=1}^{\nu'} (1 + |\alpha_i'|)^{-1} \prod_{i=\nu+1}^{l-1} (1 + |\alpha_i|)^{-1} \mu$$

where μ, μ^{-1} are bounded independent of m. The result follows from Lemma 4.9. ▮

To avoid excessive complication in the subsequent analysis, we shall assume from now on that S consists of l disconnected components, so that no point c_i lies on S and only one arc ends at each point d_i. This might be regarded as the general case, and certainly the assumption holds when all points d_i are collinear.

Suppose that we associate with each end point d_i, $i = 1,..., 2l$, a set of four points on S, $a_1^{(i)}$, $a_2^{(i)}$, $a_3^{(i)}$, $a_4^{(i)}$ not at d_i. As we travel along S from d_i, the four points are reached in the order given, and all are passed before we reach any of the points associated with the other end. Let us call the analytic Jordan arcs formed by following S from d_i to the four points, $A_1^{(i)}$, $A_2^{(i)}$, $A_3^{(i)}$, $A_4^{(i)}$.

From the form of $\phi(t)$ given by (2.3), it is clear that, near $t = d_1$, $\phi(t)$ is an analytic function of $y = (t - d_1)^{1/2}$. The variable y is suitable for use as a local variable on $\mathscr{R}$ near the point d_1. It follows, since $Z(d_1) \neq 0$, that there is a neighborhood $\mathscr{D}$ of 0 in the y-plane, in which ϕ, $X^{-(1/2)}$ are analytic and $d\phi/dy \neq 0$. The curve $\bar{S} = \{y : \operatorname{Re} \phi(y^2 + d_1) = 0\}$ is an analytic Jordan arc in the neighborhood of $y = 0$, through which it passes. If a point t_+ on one side of S corresponds to y, then t_- corresponds to $-y$.

Now suppose that $a_1^{(1)}$, $a_2^{(1)}$, $a_3^{(1)}$, $a_4^{(1)}$ are chosen so close to d_1 that they lie in the domain $\mathscr{D}$, and similarly for the other ends of S. Each of the four points is mapped into two points in the y-plane, $-b_4^{(1)}$, $-b_3^{(1)}$, $-b_2^{(1)}$, $-b_1^{(1)}$, $b_1^{(1)}$, $b_2^{(1)}$, $b_3^{(1)}$, $b_4^{(1)}$ lying on $\bar{S}$ in the order given. Let us denote an arc contained in $\bar{S}$, having end points a, b, by $\bar{S}(a, b)$. We may suppose that the $+$ side of $A_4^{(1)}$ is mapped into $\bar{S}(0, b_4^{(1)})$ and the $-$ side into $\bar{S}(0, -b_4^{(1)})$.

We assume that the weight function $\sigma(t)$ satisfies

CONDITION 6.3. Let $\sigma(t)$ be a complex function defined on S. Then

(i) there exist real A, B such that $A > |\sigma(t)| > B > 0$, $t \in S$;

(ii) for $t, t' \in S - \sum_{i=1}^{2l} A_1^{(i)}$, there exist constants L, $\lambda > 0$ such that

$$|\sigma(t')^{-1} - \sigma(t)^{-1}| < L(\ln|t' - t|)^{-1-\lambda}; \tag{6.5}$$

(iii) for $t, t' \in A_4^{(1)}$

$$|\sigma(t')^{-1} - \sigma(t)^{-1}| < L(\ln|y' - y|)^{-1-\lambda} \tag{6.6}$$

and similarly at the other ends.

We show in Appendix 2 that these conditions imply a fourth.

(iv) There is a polynomial $\rho_m(t) = \sigma_m^{-1}(t)$ of degree $m - l + 1$ such that for large m

$$\sup_{t \in S} |\sigma(t) - \sigma_m(t)| < \text{const}(\ln m)^{-1-\lambda}.$$

From now on, we use the sequence of weight functions σ_m given in Condition 6.3.

Several lemmas needed later now follow.

LEMMA 6.4. *Let us define in S'*

$$\chi_m(t) = \exp\left(\frac{X^{1/2}(t_1)}{2\pi i}\int_S dt'\,(t'-t)^{-1}\,X_{+}^{-(1/2)}(t')\ln\sigma_m(t')\right) \tag{6.7}$$

and $\chi(t)$ in a corresponding way. Then, for $t\in S$, the limits $\chi_+(t), \chi_-(t)$ exist and $\chi_m(t)\to\chi(t)$ uniformly as $m\to\infty$, for $t\in S'$ and $\chi_{m\pm}(t)\to\chi_\pm(t)$, uniformly, $t\in S$.

Proof. The proof is analogous to that of Szegö's [16]. Let us suppose that we wish to study the limit as t approaches a point $t_0\in A_2^{(1)}$ on the $+$ side of S. Then we consider

$$I = X^{1/2}(t_1)\int_{A_3^{(1)}} dt'\,(t'-t)^{-1}\,X_{+}^{-(1/2)}(t')\ln\sigma(t') \tag{6.8}$$

for the contribution to the integral in (6.7) from $S - A_3^{(1)}$ obviously has a limit as $t\to t_{0+}$. Define $s(y) = \ln\sigma(t)$ and $\bar{X}(y) = X(t)(t-d_1)^{-1}$. Then I can be written

$$I = 2\bar{X}^{1/2}(y)\,y\int_{S(0,b_3^{(1)})} dy'\,(y'^2-y^2)^{-1}\,\bar{X}^{1/2}(y')\,s(y')$$

and if we set $s(-y) = s(y)$, we have

$$I = \bar{X}^{1/2}(y)\int_{S(-b_3^{(1)},b_3^{(1)})} dy'\,(y'-y)^{-1}\,\bar{X}^{1/2}(y')\,s(y'). \tag{6.9}$$

Since from (6.6) $s(y)$ satisfies, for $y, y'\in\bar{S}(-b_3^{(1)}, b_3^{(1)})$,

$$|\,s(y)-s(y')| < \mathrm{const}(\ln|\,y-y'\,|)^{-1-\lambda} \tag{6.10}$$

a standard argument shows that the limit of I exists at $t\to t_{0+}$.

To discuss convergence for large m, still with $t_0\in A_2^{(1)}$, we again need to study only

$$\begin{aligned} I_m &= \bar{X}_1^{1/2}(t)\int_{A_3^{(1)}} dt'\,(t'-t)^{-1}\,X_{+}^{-(1/2)}(t')\ln\sigma_m(t') \\ &= \bar{X}^{1/2}(y)\int_{S(-b_3^{(1)},b_3^{(1)})} dy'\,(y'-y)^{-1}\,\bar{X}^{-1/2}(y')\,s_m(y') \end{aligned} \tag{6.11}$$

with $s_m(y) = s_m(-y) = \ln\sigma_m(t)$.

Following Szegö, we split $\bar{S}(-b_3^{(1)}, b_3^{(1)})$ into an interval E containing those points with distance from y_0 no greater than $m^{-1}(\ln m)^{-\lambda}$, and its complement E'. For large enough m, $E \subset \bar{S}(-b_3^{(1)}, b_3^{(1)})$. We have

$$\begin{aligned} \bar{X}^{(1,2)}(y)(I_m - I) &= \int_{E'} dy'(y'-y)^{-1}\bar{X}^{(1,2)}(y')(s_m(y') - s(y')) \\ &\quad + (s_m(y) - s(y)) \int_F dy'(y'-y)^{-1}\bar{X}^{(1,2)}(y') \\ &\quad + \int_E dy'(y'-y)^{-1}\bar{X}^{(1,2)}(y')[(s_m(y') - s_m(y)) \\ &\quad - (s(y') - s(y))]. \end{aligned} \tag{6.12}$$

Now $\bar{S}(-b_4^{(1)}, b_4^{(1)})$ is an analytic Jordan arc, and $\rho_m(y^2 - d_1)$ is a polynomial in y of degree $2m$. A generalization of the extension of Bernstein's theorem proved by Widom [17] shows that, for $y \in \bar{S}(-b_3^{(1)}, b_3^{(1)})$,

$$|d\rho_m/dy| \le \text{const}\, m \sup_{u \in \bar{S}(-b_4^{(1)}, b_4^{(1)})} |\rho_m|. \tag{6.13}$$

It follows that

$$|s_m(y') - s_m(y_0)| < \text{const}\, m\, |y' - y_0|, \qquad y' \in E.$$

Szegö's argument now applies to (6.12) to give the required result.

If t is near another end, the argument is similar, while if $t \in S - \sum_{i=1}^{2l} A_2^{(1)}$, a analogous argument applies, without the need for a transformation of variable. ∎

Let us define the function $H(x, t)$, t on the first sheet of $\mathscr{R}$ by

$$\begin{aligned} H_1(x,t) &= \theta(x) \exp\left\{\frac{X^{1,2}(t_1)}{2\pi i} \int_S dt'(t'-t)^{-1} X_+^{(1,2)}(t')[2\ln(t'-d_1) \right. \\ &\quad \left. - \ln(t'-x)] + \ln(t-x) - \ln(t-d_1)\right\} \qquad x \text{ on first sheet}, \end{aligned} \tag{6.14}$$

$$\begin{aligned} H_2(x,t) &= \theta(x) \exp\left\{\frac{X^{1,2}(t_1)}{2\pi i} \int_S dt'(t'-t)^{-1} X_+^{(1,2)}(t') \ln(t'-x)\right\} \\ &\qquad x \text{ on second sheet}. \end{aligned}$$

In the integrals, the logarithms are determined as in Lemma 5.2.

LEMMA 6.5. (i) *The function $H(x, t)$ is uniformly bounded for $t \in S$, independent of x. There exists a neighborhood $\mathscr{S}'$ of $y = 0$, $\mathscr{S}' \subset \mathscr{S}$, such that for $y \in \mathscr{S}'$, $H(x, y^2 - d_1)$ is an analytic function of y with derivative uniformly bounded, independent of x, and similarly near the other ends of S.*

(ii) *For t in a closed bounded domain of S′, $H(\alpha, t)$ is a continuous function of α on the dissected $\mathscr{R}$, using the topology provided by the local coordinate. (The dissection of $\mathscr{R}$ referred to makes $\mathscr{R}$ simply connected in such a way that the path from ∞_2 to α is that used in the definition of $u_k(\alpha)$.)*

Proof. (i) For $t \in S - \sum_{i=1}^{2l} A_2^{(1)}$, the boundedness follows immediately, if necessary with the help of a contour distortion.

Suppose then that $y = (t - d_1)^{1/2} \in \mathscr{D}$. In a way analogous to that in which we obtained (5.10) we see, for α on the first sheet, that

$$H_1(\alpha, t) = \theta(\alpha) \exp \Big\{ \tfrac{1}{2} X^{1/2}(t_1) \Big(\int_{\infty}^{\alpha} dt' \, (t' - t)^{-1} X^{-(1/2)}(t_1') - 2 \int_{\infty}^{d_1} dt' \, (t' - t)^{-1} X^{-(1/2)}(t_1') \Big) + \tfrac{1}{2} \ln(t - \alpha) \Big\} . \quad (6.15)$$

For $\mathscr{D}'$ choose a neighborhood of $y = 0$ with boundary a positive distance from that of $\mathscr{D}$. Let $\beta \in S'$ be such that $y_0 = (\beta - d_1)^{1/2} \in \mathscr{D} - \mathscr{D}'$ and y_0 is at a positive distance from $\mathscr{D}'$.

If α is not near any end of S the contribution in (6.15) from $\int_{\infty}^{\alpha}$ obviously satisfies the statement of the lemma. Suppose then that $z = (\alpha - d_1)^{1/2}$ and $z \in \mathscr{D}$. We write $\int_{\infty}^{\alpha} = \int_{\infty}^{\beta} + \int_{\beta}^{\alpha}$, and need only consider $\int_{\beta}^{\alpha}$.

We have, using $t' = y'^2 + d_1$,

$$\begin{aligned} &\tfrac{1}{2} X^{1/2}(t_1) \int_{\beta}^{\alpha} dt' \, (t' - t)^{-1} X^{-(1/2)}(t_1') + \tfrac{1}{2} \ln(t - \alpha) \\ &= \bar{X}^{1/2}(y) \, y \int_{y_0}^{z} dy' \, (y'^2 - y^2)^{-1} \bar{X}^{-(1/2)}(y') + \tfrac{1}{2} \ln(y^2 - z^2) \\ &= \bar{X}^{-(1/2)}(y) \, y \int_{y_0}^{z} dy' \, (y'^2 - y^2)^{-1} (\bar{X}^{-(1/2)}(y') - \bar{X}^{-(1/2)}(y)) \\ &\quad + y \int_{y_0}^{z} dy' \, (y'^2 - y^2)^{-1} + \tfrac{1}{2} \ln(y^2 - z^2). \end{aligned} \quad (6.16)$$

The analyticity of $\bar{X}^{-(1/2)}(y')$ in y'^2 shows that the first term in (6.16) is analytic in y, while the remaining terms give

$$\ln(y - z) + \tfrac{1}{2}(\ln(y_0 + y) - \ln(y_0 - y)) + \text{const.}$$

The treatment when α is near another end of S is similar.

The term $\int_{\infty}^{d_1}$ may be treated in the same way, but now we have

$$X^{1/2}(t_1) \int_{\beta}^{d_1} dt' \, (t' - t)^{-1} X^{-(1/2)}(t_1') = (\text{analytic in } y).$$

The case of large α is treated as in Lemma 6.2, and the discussion when α is on the second sheet is similar to the above.

(ii) The continuity is immediate except when α changes sheets, which it can only do by crossing the arc of S ending at d_1. We must therefore show that

$$H_1(\alpha_+, t) = H_2(\alpha_-, t)$$

for α on the arc of S ending at d_1.

Now we see that

$$\int_x^{\alpha_+} dt'\,(t'-t)^{-1} X^{-(1/2)}(t_1') - 2\int_x^{d_1} dt'\,(t'-t)^{-1} X^{-(1/2)}(t_1')$$

$$= \int_{d_1}^{\alpha_+} dt'\,(t'-t)^{-1} X^{-(1/2)}(t_1') - \int_x^{d_1} dt'\,(t'-t)^{-1} X^{-(1/2)}(t_1')$$

$$= -\int_{d_1}^{\alpha_-} dt'\,(t'-t)^{-1} X^{-(1/2)}(t_1') - \int_x^{d_1} dt'\,(t'-t)^{-1} X^{-(1/2)}(t_1')$$

$$= -\int_x^{\alpha_-} dt'\,(t'-t)^{-1} X^{-(1/2)}(t_1').$$

Using (6.15) and the equivalent formula for $H_2(\alpha, t)$,

$$H_2(\alpha, t) = \theta(\alpha) \exp\left\{-\tfrac{1}{2}X^{1/2}(t_1)\int_x^{\alpha} dt'\,(t'-t)^{-1} X^{-(1/2)}(t_1') + \tfrac{1}{2}\ln(t-x)\right\},$$

the result follows. ∎

LEMMA 6.6. *There is a neighborhood of* $\bar{S}(-b_4^{(1)}, b_4^{(1)})$ *in which* $\chi_m(y^2 + d_1)$ *is an analytic function of* y. *For* $y \in \bar{S}(-b_2^{(1)}, b_2^{(1)})$

$$\left|\frac{d\chi_m(y^2+d_1)}{dy}\right| < \text{const}\ m \ln m$$

for large enough m, *where the constant is independent of* m.

Proof. If we substitute $\sigma_m^{-1}(t) = \rho_0 \prod_{i=1}^{m} (t - r_i)$ into (6.7) we find

$$\chi_m(t) = \exp\left\{\frac{X^{1/2}(t_1)}{2\pi i}\int_S dt'\,(t'-t)^{-1} X_+^{-(1/2)}(t')\left[\ln \rho_0 + \sum_{i=1}^{m} \ln(t - r_i)\right]\right\}.$$

Lemma 6.5, along with a similar result for the $\ln \rho_0$ term, gives the analyticity.

To discuss the derivative, we follow the approach used in Lemma 6.4, and the problem quickly reduces to obtaining a bound for the derivative of I_m of (6.11). This means that we need a bound for the derivative of

$$\int_{\bar{S}(-b_3^{(1)}, b_3^{(1)})} dy'\,(y'-y)^{-1} \bar{X}^{-(1/2)}(y')\, s_m(y'). \tag{6.17}$$

In this expression, we are to take the limit as y approaches $\bar{S}(-b_2^{(1)}, b_2^{(1)})$ from one side. The function $s_m(y')$ is analytic in a neighborhood of $\bar{S}(-b_3^{(1)},$

$b_3^{(1)}$), so that (6.17) is analytic in y for y in a neighborhood of $\bar{S}(-b_2^{(1)}, b_2^{(1)})$. We shall differentiate (6.17) and then take the limit as $y \to \bar{S}(-b_2^{(1)}, b_2^{(1)})$.

Differentiating with respect to y and then integrating by parts gives from (6.17)

$$\int_{\bar{S}(-b_3^{(1)},b_3^{(1)})} dy'\,(y'-y)^{-1}\left(s_m(y')\,\frac{d(\bar{X}^{-(1/2)})}{dy'}+\bar{X}^{-(1/2)}(y')\,\frac{ds_m}{dy'}\right)$$
$$-\,(y'-y)^{-1}\,\bar{X}^{-(1/2)}(y')\,s_m(y')\Big|_{-b_3^{(1)}}^{b_3^{(1)}}. \tag{6.18}$$

The arguments of Lemma 6.4 give the required bound except for the term

$$\int_{\bar{S}(-b_3^{(1)},b_3^{(1)})} dy'\,(y'-y)^{-1}\,\bar{X}^{-(1/2)}(y')(ds_m/dy'). \tag{6.19}$$

We see from (6.13) that

$$ds_m/dy < \text{const} \qquad \text{for } y \in \bar{S}(-b_3^{(1)}, b_3^{(1)}). \tag{6.20}$$

In the same way as in Lemma 6.4, using the generalization of Bernstein's theorem on $(d/dy)(\rho_m(y^2+d_1))$, we find

$$|\,d^2s_m/dy^2\,| < \text{const}\; m^2 \qquad \text{for } y \in \bar{S}(-b_3^{(1)}, b_3^{(1)}),$$

so that

$$\left|\frac{ds_m}{dy'}-\frac{ds_m}{dy}\right| < \text{const}\; m^2\,|\,y'-y\,|, \qquad y, y' \in \bar{S}(-b_3^{(1)}, b_3^{(1)}). \tag{6.21}$$

Now write (6.19) as

$$\int_{\bar{S}(-b_3^{(1)},b_3^{(1)})} dy'\,(y'-y)^{-1}\,\bar{X}^{-(1/2)}(y')\,\frac{ds_m}{dy'}$$
$$=\frac{ds_m}{dy}\int_{\bar{S}(-b_3^{(1)},b_3^{(1)})} dy'\,(y'-y)^{-1}\,\bar{X}^{-(1/2)}(y')$$
$$+\int_{\bar{E}} dy'\,(y'-y)^{-1}\,\bar{X}^{-(1/2)}(y')\left(\frac{ds_m}{dy'}-\frac{ds_m}{dy}\right)$$
$$+\int_{\bar{E}'} dy'\,(y'-y)^{-1}\,\bar{X}^{-(1/2)}(y')\left(\frac{ds_m}{dy'}-\frac{ds_m}{dy}\right)$$

where $\bar{E}=\{y' : y' \in \bar{S}(-b_3^{(1)}, b_3^{(1)}),\ |\,y'-y\,| \leqslant m^{-1}\}$ and $\bar{E}'$ is its complement in $\bar{S}(-b_3^{(1)}, b_3^{(1)})$.

The limit as $y \to \bar{S}(-b_2^{(1)}, b_2^{(1)})$ now obviously exists and the bounds (6.20) and (6.21) give the required result. ∎

It is necessary to relate $F(t)$ to the function $\phi(t)$ of Section 2. This is done by the following, LEMMA 6.7. *The function $\phi(t)$ may be written on the first sheet as $\phi(t) = \Phi_1(t)$,*

$$\Phi_1(t) = \phi_0 + X^{1/2}(t_1)\Big\{(1/\pi i)\int_S dt'\,(t'-t)^{-1} X_+^{-(1/2)}(t')\ln(t'-d_1)$$
$$+ \sum_{j=1}^{2(l-1)} b_j \int_{L_j} dt'\,(t'-t)^{-1} X^{-(1/2)}(t_1')\Big\} - \ln(t-d_1), \qquad (6.22)$$

where ϕ_0 is pure imaginary, and b_j is defined in Lemma 4.1. In the integral, the branch of $\ln(t'-d_1)$ *is chosen as in Lemma 5.2.*

Proof. The function $\Psi(t)$, given by

$$\Psi(t) = (1/\pi i)\int_S dt'\,(t'-t)^{-1} X_+^{-(1/2)}(t')$$

is analytic and single-valued in S'. For $t \in S$, Plemelj's formula [15] gives

$$\Psi_+(t) - \Psi_-(t) = 2X_+^{-(1/2)}(t)\ln(t-d_1).$$

Thus we have

$$-X_+^{1/2}(t_1)\,\Psi_+(t) - \ln(t-d_1) = -X_-^{1/2}(t_2)\,\Psi_-(t) - \ln(t-d_1),$$

and it follows that

$$\Phi_2(t) = \phi_0 - X^{1/2}(t_2)\Big\{\Psi(t) - \sum_{j=1}^{2(l-1)} b_j \int_{L_j} dt'\,X^{-(1/2)}(t_1')(t'-t)^{-1}\Big\}$$
$$- \ln(t-d_1)$$

gives a continuation of $\Phi(t)$ onto the second sheet of $\mathscr{R}$. If this function is continued across a possibly different arc of S, the original function $\Phi(t)$ results.

This discussion assumes that no arc L_j has been crossed during the continuation. If this should happen, $\pm 2\pi i b_j$ must be added for each arc L_j crossed. The conclusion is that $\Phi(t)$ is a multivalued function defined on $\mathscr{R}$, any determination of which having on sheet i, $i = 1, 2$ the form $\Phi_i(t)$ + pure imaginary.

As in Lemma 5.2, the coefficient of t^{-k} in an expansion of the integrals in brackets in (6.22) for large t is

$$-(1/\pi i)\int_S dt'\,X_+^{-(1/2)}(t')\,t'^{k-1}\ln(t'-d_1) - \sum_{j=1}^{2(l-1)} b_j \int_{L_j} dt'\,X^{-(1/2)}(t_1')\,t'^{k-1}$$
$$= \int_{\infty}^{d_1} dt'\,X^{-(1/2)}(t_1')\,t'^{k-1} - \tfrac{1}{2}\sum_{j=1}^{2(l-1)} b_j\Omega_{kj}$$
$$= \tfrac{1}{2}u_k(\infty_1) - \tfrac{1}{2}\sum_{j=1}^{2(l-1)} b_j\Omega_{kj},$$

and this is zero for $k < l$ from (4.7). Thus, as $t \to \infty$, $\Phi_1(t) \sim \ln t + \text{const}$, $\Phi_2(t) \sim -\ln t + \text{const}$.

The methods of the previous lemmas may be used to show that $\Phi(t)$ is analytic on $\mathscr{R}$ in the local variable $(t - d_i)^{1/2}$ near $t = d_i$. It also may be deduced that $\Phi(t) - \phi_0$ is pure imaginary at $t = d_1$.

The difference between $\Phi(t)$ and $\phi(t)$ is a function analytic on $\mathscr{R}$, having pure imaginary periods, and the only such function is a constant [8]. The fact that $\phi(d_1) = 0$ completes the proof. ∎

LEMMA 6.8. *Suppose that $r_m(t)$ is any polynomial in t of degree m, such that* $\sup_{t\in S} | r_m(t)| = 1$. *Define the degree m polynomial $R_m(t)$ by*

$$R_m(t) = \mu_m^{-1} \int_S dt' \, X_+^{-(1/2)}(t')(\sigma(t') - \sigma_m(t'))\, K(t, t')(t' - t)^{-1}\, r_m(t'). \tag{6.23}$$

Then, if m is large enough, for $m \in \Sigma$ we have

$$| R_m(t)| < \text{const}(\ln m)^{-\gamma} \tag{6.24}$$

where the constant is independent of m and r_m.

Proof. The first step is to prove that F, F' are uniformly bounded on S. From (5.7) and (6.22), we have

$$F(t) = e^{n\phi(t)} \prod_{i=1}^{l-1} H(\alpha_i, t)\, \chi_m(t) \exp - \left\{ X^{1/2}(t_1) \sum_{j=1}^{2(l-1)} \xi_j \int_{L_j} dt' \, (t' - t)^{-1} X^{-(1/2)}(t_1') - n\phi_0 \right\} \tag{6.25}$$

and the boundedness follows from Lemmas 4.1, 5.4, 6.4, 6.5, and the fact that $\operatorname{Re} \phi = 0$ on S.

If $t \in S - \sum_{i=1}^{2l} A_2^{(i)}$, Szegö's method applies almost directly. Since $K(t, t')$ is a polynomial in t' of degree m, the generalization of Bernstein's theorem used in Lemma 6.4 shows that, for $t, t' \in S - \sum_{i=1}^{2l} A_1^{(i)}$

$$| K(t, t')| < \text{const}\, m \mid t - t' \mid \sup_{t' \in S} | K(t, t')| . \tag{6.26}$$

The definition (6.2) of $K(t, t')$ shows that its modulus is uniformly bounded for $t, t' \in S$. The result (6.24) follows after splitting S into a part with points distance $\leqslant m^{-1} (\ln m)^{-1}$ from t, and its complement, just as in Szegö.

Now suppose that $t \in A_2^{(1)}$. The contribution to $R_m(t)$ from the integral over $S - A_3^{(1)}$ may be bounded, using (6.25), by

$$\mu_m^{-1}(\ln m)^{-1-\lambda} \text{const} \int_{S - A_3^{(1)}} dt' \, X_+^{-(1/2)}(t')$$

which satisfies (6.24).

To treat the remainder of the integral, we shall need suitable bounds for $K(t, t')$, which is

$$K(t, t') = (F_+(t) + F_-(t))(F_+'(t') + F_-'(t')) - (F_+'(t) + F_-'(t))(F_+(t') + F_-(t')). \tag{6.27}$$

If we write

$$F(t) = e^{m\phi(t)}\chi_m(t)\, g(t) \tag{6.28}$$

then one part of (6.27) becomes

$$\begin{aligned} F_+(t)\, F_+'(t') - F_+'(t)\, F_+(t') &= e^{m(\phi_+(t)+\phi_+(t'))}\chi_{m+}(t)\, \chi_{m+}(t') \\ &\quad \times \{e^{\phi_+(t')} g_+(t)\, g_+'(t') - e^{\phi_+(t)} g_+'(t)\, g_+(t')\}. \end{aligned} \tag{6.29}$$

From Lemma 6.5 and a similar result for the integral over L_j in (6.25) we see that the expression in brackets in (6.29) is analytic in y' for $y' \in \mathscr{D}$, with bounded derivative, and vanishes at $y' = y$. Since the other factors in (6.29) are bounded on S, we deduce for $y, y' \in \mathscr{D}$,

$$|F_+(t)\, F_+'(t') - F_+'(t)\, F_+(t')| < \text{const}\, |y' - y|\, .$$

A similar argument applies to another part of (6.27), so that we write, for $y, y' \in \mathscr{D}$

$$\begin{aligned} F_+(t)\, F_+'(t') - F_+'(t)\, F_+(t') &+ F_-(t)\, F_-'(t') - F_-'(t)\, F_-(t') \\ &= (y - y')\{e^{m(\phi_+(t)+\phi_+(t'))}\chi_{m+}(t)\, \chi_{m+}(t')\, G(y, y') \\ &\quad + e^{m(\phi_-(t)+\phi_-(t'))}\chi_{m-}(t)\, \chi_{m-}(t')\, G(-y, -y')\} \end{aligned} \tag{6.30}$$

where $G(y, y')$ is analytic in y, y' and

$$e^{\phi(t')}g(t)\, g'(t') - e^{\phi(t)}g'(t)\, g(t') = (y - y')\, G(y, y').$$

Since the left-hand side of (6.30) vanishes if $t = t'$, it follows that the term in brackets on the right-hand side of (6.30) vanishes when $y' = -y$. We use Lemma 6.6 to bound the derivative of χ_m with respect to y' by const m. The other terms in the bracketed factor in (6.30) have a similar bound. We may therefore conclude that the bracketed factor is bounded by const $|y + y'|\, m \ln m$ for $y, y' \in \bar{S}(-b_2^{(1)}, b_2^{(1)})$.

The rest of $K(t, t')$ may be treated as above, so that we end up with the two bounds for $y, y' \in \bar{S}(-b_2^{(1)}, b_2^{(1)})$

$$|K(t, t')| < \text{const}(|y' - y| + |y' + y|) \tag{6.31}$$

$$< \text{const}\, |y' - y|\, |y' + y|\, m \ln m. \tag{6.32}$$

We now have for $t \in A_2^{(1)}$

$$\left|\int_{A_4^{(1)}} dt'\, X_+^{-(1/2)}(t')(\sigma(t) - \sigma_m(t'))\, K(t, t')(t' - t)^{-1}\, r_m(t')\right|$$
$$< \text{const}\, (\ln m)^{-1-\lambda} \int_{S(0, b_2^{(1)})} |\, dy' \,||\, y'^2 - y^2 \,|^{-1}\, |\, K(t, t')| \,. \quad (6.33)$$

The integration contour in (6.33) is split into a part for which $|\, y' - y \,| < m^{-1}$, in which bound (6.32) is used, and its complement in $\bar{S}(0, b_2^{(1)})$, where bound (6.31) is used. The result follows immediately. ∎

We come now to a basic theorem.

THEOREM 6.9. *Suppose that S has l components and that $\sigma(t)$ satisfies Condition 6.3. Then, provided $m \in \Sigma$ is large enough, $p_m(t)$, the orthogonal polynomial of order m for weight $\sigma(t)$, is unique and may be normalized so that*

$$p_m(t) = q(t) + (\ln m)^{-\lambda}\, e^{m\phi(t)} \gamma_m(t) \quad (6.34)$$

where $\gamma_m(t)$ is uniformly bounded in any bounded region of the complex plane.

Proof. It follows from Lemma 6.8 that, for large enough m, integral equation (6.1) can be solved uniquely by iteration. There is also no solution of the corresponding homogeneous equation so that the integral in (6.3) cannot be zero, and $p_m(t)$ is unique up to a constant factor.

The form (6.34) follows immediately from our previous results. ∎

7. CONVERGENCE OF PADÉ APPROXIMANTS

For the values of m in the sequence Σ used in Theorem 6.9, the convergence in capacity of the diagonal Padé approximants to $f(t)$ for t in a closed bounded domain not containing S follows immediately from the result of the theorem and (5.6). For the other integers, it follows from Lemma 3.4, that only the basic integers need be considered.

LEMMA 7.1. *There exists an infinite sequence Σ' of basic integers and an integer $m_0 > 0$ with the following properties.*

(i) *If $m \in \Sigma, m > m_0$, and ν is the largest integer in Σ' that satisfies $\nu \leqslant m$, then $p_m(t) = r(t)\, p_\nu(t)$ with some polynomial $r(t)$ of degree $\leqslant l$.*

(ii) *The difference between two consecutive integers in Σ' does not exceed $2l$.*

Proof. (i) From Theorem 6.9 it follows that for each $m \in \Sigma$, greater than some m_0, the orthogonal polynomial of order m, $p_m(t)$ is essentially unique.

From Lemma 3.4(ii), then, either m or $(m + 1)$ is a basic integer. Let $\Sigma' = \{\nu : \nu > m_0, \nu \text{ is basic}, \nu \text{ or } \nu + 1 \in \Sigma\}$. Since the consecutive members in Σ' differ by no more than l, m and ν of the lemma satisfy the relation $m - \nu \leq l$. The polynomial $p_m(t)$ can now be written as asserted in the lemma and the degree of $r(t)$ cannot exceed l.

(ii) Follows from the fact that if m, ν are as in (i) then $m - \nu \leq l$. ∎

LEMMA 7.2. *For each basic integer $\nu \geq m_0 + 8l$, there exist $m, m^* \in \Sigma'$ with $m^* > m$, $|m - \nu| \leq 2l$, and polynomials Q, Q^*, D, of degree $\leq 8l$, such that*

$$p_\nu(t) = \frac{Q(t)\, p_m(t) + Q^*(t)\, p_{m^*}(t)}{D(t)}. \tag{7.1}$$

$D(t) \neq 0$ for some t.

Proof. The proof follows from Corollary 3.6 and Lemma 7.1. ∎

Let $R \subset S'$ be a closed, simply connected, bounded domain, and L_j be so chosen that they do not intersect R. This choice of L_j does not alter the results established so far. Since Re $\phi(t)$ is the Green's function, it may be shown [11] that the locii Re $\phi(t) = \lambda$ for different positive λ are nested so that every point t for which Re $\phi(t) = \lambda$ is contained within a closed curve making up part of the locus Re $\phi(t) = \lambda'$ provided that $\lambda < \lambda'$.

Define M, M' by

$$M = \inf_{t \in R} (\operatorname{Re} \phi(t)), \quad M' = \sup_{t \in R} (\operatorname{Re} \phi(t))$$

and denote by R' the closed, bounded domain containing R given by

$$R' = \{t : M/10l \leq \operatorname{Re} \phi(t) \leq M'\}.$$

For each basic integer ν of Lemma 7.2 let $F_m(t)$, $F_m{}^*(t)$ be the functions (5.6) corresponding to weight function $\rho_m^{-1}(t)$ and values of $n = m, m^*$, respectively. As in (6.28) we write

$$F_m = e^{m\phi} \chi_m g,$$

$$F_m{}^* = e^{m^*\phi} \chi_m g^*.$$

We normalize $p_\nu(t)$ by requiring that

$$D(t) = \prod (t - t_j)$$

and

$$\sup_{t \in R} [|Q(t)|^2 + |Q^*(t)|^2] = 1. \tag{7.2}$$

LEMMA 7.3. *Let $\eta_m(t) = Q(t)\, g(t) + e^{\Delta\phi(t)} Q^*(t)\, g^*(t)$, $\Delta = m^* - m$, and a determination of g, g^* is chosen to make $\eta_m(t)$ single-valued in R and η_m*

$\chi_m^{-1} e^{-m\phi}(QF_m + Q^*F_m^*)$. *Then given* $\mu > 0$, *for sufficiently large* m, *there exists* $\delta_0(\mu) > 0$ *such that* $|\eta_m(t)| > \delta_0(\mu)$ *for all* $t \in R - R_\mu$ *where* $R_\mu \subset R$ *and* $\operatorname{Cap}(R_\mu) \leqslant \mu$.

Proof. First we show that for each m, $\eta_m(t) \neq 0$ for some $t \in R$. If this was not the case, then one would have that

$$\frac{F_m(t)}{F_m^*(t)} = -\frac{Q^*(t)}{Q(t)}. \tag{7.3}$$

Equation (7.3) implies the relation among divisors

$$\frac{\infty_1^{-m}\alpha_m r_1 \cdots r_{\bar{m}}}{\infty_1^{-m^*}\alpha_m^* r_1 \cdots r_{\bar{m}} \infty_2^{m^*-m}} = \frac{\beta_1 h(\beta_1) \cdots \beta_i h(\beta_i)\, \infty_1^{-i}\infty_2^{-i}}{\gamma_1 h(\gamma_1) \cdots \gamma_j h(\gamma_j)\, \infty_1^{-j}\infty_2^{-j}} \tag{7.4}$$

where α_m, α_m^* are the integral divisors to be used in (5.1) corresponding to F_m, F_m^*, and $\beta_1, \ldots, \beta_i$, $\gamma_1, \ldots, \gamma_j$ are the zeros of Q^*, Q in the complex plane. We assume that all common factors of Q^*, Q have been removed, and hence $\beta_i \neq \gamma_j$, any i, j. Equation (7.4) is equivalent to

$$(\infty_1)^{\Delta+j+i}\, \alpha_m \gamma_1 h(\gamma_1) \cdots \gamma_j h(\gamma_j) = \beta_1 h(\beta_1) \cdots \beta_i h(\beta_i)\, \alpha_m^*(\infty_2)^{\Delta-j+1}. \tag{7.5}$$

Since α_m, α_m^* have no pairs, the only possible way for (7.5) to hold is that $i = j = 0$, which implies that

$$(\infty_1)^{\Delta}\, \alpha_m = \alpha_m^*(\infty_2)^{\Delta}. \tag{7.6}$$

This means that α_m has ∞_2 for some of its components, which is impossible from Lemma 4.9. Thus for each m, $\eta_m(t) \neq 0$ for some $t \in R$.

Further, for each m, $\eta_m(t)$ is an analytic function of $t \in R$. Therefore $\operatorname{Cap}\{t : |\eta_m(t)| = 0\} = \xi$ for any $\xi > 0$ implies that $\eta_m(t) = 0$ for all $t \in R$. Since $\eta_m(t)$ is not identically equal to zero, given $\mu > 0$, for each m, one can find $\delta_m(\mu) > 0$ such that

$$\operatorname{Cap}\{t : |\eta_m(t)| \leqslant \delta_m(\mu)\} = \mu.$$

Now, suppose that the result of the lemma were not true. Then one could find a subsequence $\{k\}$ such that $\delta_k(\mu) \to_{k\to\infty} 0$. The sequence $\{k\}$ may be so chosen that $m^* - m = \Delta$ is fixed. In the following, we show that this leads to a contradiction.

Let a_k and a_k^* denote the set of the coefficients of the powers of t appearing in Q, Q^*, respectively. Consider the point $P^k = \alpha_k \alpha_k^* a_k a_k^* \in \mathscr{S} \subset \mathscr{R}^{2(l-1)} \times C^{16l}$ where $\mathscr{R}^{2(l-1)}$ is the Cartesian product of $2(l-1)$ identical copies of $\mathscr{R}$. Since the canonically dissected $\mathscr{R}$ is compact, and owing to condition (7.2), a_k, a_k^* are confined to a bounded subset of C^{8l}, $\mathscr{S}$ can be chosen to be compact. Hence there is a subsequence of the values of $\{k\}$, which we still denote by $\{k\}$, such that $P^k \to_{k\to\infty} P \in \mathscr{S}$.

The functions $Q(t)$, $Q^*(t): C^{8l} \times R \to C$ are continuous on C^{8l} and analytic on R. Also since $\xi_j(\alpha_k)$, $\xi_j(\alpha_k^*)$ are bounded (Lemma 4.1), they contain a convergent subsequence. Denoting this subsequence by $\{k\}$ again, and noticing that $H(\alpha, t)$ is continuous on $\mathscr{S}$ and analytic on R, we have from (6.25), that $F_k(t)$, $F_k^*(t)$: $\mathscr{S} \times R \to C$ map a convergent sequence in $\mathscr{S}$ into a convergent one.

Thus $\eta_k(t) = \eta(P^k, t): \mathscr{S} \times R \to C$, for each k, is continuous on $\mathscr{S}$ and analytic on R. And since $P^k \to P$, $\eta_k(t) \to \eta_0(t) = \eta(P, t)$. The compactness of R, together with the continuity of $\eta_k(t)$ for $t \in R$, implies that the convergence is uniform with respect to $t \in R$ and hence $\eta_0(t)$ is analytic in R.

Further, since $\delta_k(\mu) \to_{k\to\infty} 0$, we have that

$$\operatorname{Cap}\{t: \eta_0(t) \neq 0\} \leqslant \mu \to 0$$

and hence, because of the analyticity of $\eta_0(t)$, $\eta_0(t) = 0$ for all $t \in R$.

Also since $P^k \to P$, a_k, $a_k^* \to a_0$, a_0^*, α_k, $\alpha_k^* \to \alpha_0$, α_0^*,

$$Q(t), Q^*(t) \to Q_0(t), Q_0^*(t) \qquad \text{and} \qquad g(t), g^*(t) \to g_0(t), g_0^*(t).$$

We have used that $\chi_k^{-1}(t)$ has a limit. Since $Q(t)$, $Q^*(t) \to Q_0(t)$, $Q_0^*(t)$, β_j, $\gamma_j \to \beta_j^0, \gamma_j^0$ for each j. Since $\{\{\gamma_j^0\}\alpha_0\}$ is a finite set, one can find a set $R_{\mu/2} \subset R$, such that $\operatorname{Cap}(R_{\mu/2}) \leqslant \mu/2$ and for large enough k, $\{\{\gamma_j\}\{\gamma_j^0\}\alpha_0\alpha_k\} \subset R_{\mu/2}$. Now

$$\begin{aligned}
\eta_k(t) &= Q(t)\, g(t) \left\{1 + e^{2i\phi(t)} \frac{Q^*(t)\, g^*(t)}{Q(t)\, g(t)}\right\} \\
&\to Q_0(t)\, g_0(t) \left\{1 + \frac{e^{2i\phi(t)} Q_0^*(t)\, g_0^*(t)}{Q_0(t)\, g_0(t)}\right\} \\
&\qquad \eta_0(t) = 0 \qquad \text{for } t \in R'_{\mu/2}
\end{aligned} \tag{7.7}$$

where $R'_{\mu/2} \subset R$ is some set with $\operatorname{Cap}(R'_{\mu/2}) \leqslant \mu/2$. It is obvious now, that

$$\frac{F_0(t)}{\bar{F}_0(t)} = -\frac{Q_0^*(t)}{Q_0(t)} \qquad \text{for } t \in R'_{\mu/2}. \tag{7.8}$$

Equation (7.8) implies the relation (7.4) among divisors, with α_m, α_m^* replaced by α_0, α_0^* on the left and the β_j, γ_j replaced by β^0, γ_j^0 on the right. The same argument leads to the same contradiction as in the case of (7.4). ∎

LEMMA 7.4. *The polynomial p_ν of* (7.1) *has the following properties.*

(i) *Given $\mu > 0$, there exists $\nu_0(\mu)$ such that, for each $\nu > \nu_0$*

$$|p_\nu(t)|^{-1} \leqslant \text{const}\, d\, e^{-\nu M} \delta_0^{-1}(\mu)$$

for all $t \in R$ except for a set of capacity $\leqslant \mu$, with $d = \sup_{t\in R} |D(t)|$.

(ii) *There is a constant independent of* ν *such that*

$$|p_\nu(t)| \leqslant \text{const } e^{(9/10)mM}$$

for all $t \in S$.

Proof. (i) Using (5.11) and (5.6), it is not difficult to show that, for $t \in R$,

$$|F_m(h(t))| < \text{const } e^{-mM}.$$

Thus, for $t \in R$, we have, with (6.34)

$$p_m(t) = F_m(t) + O(e^{-mM}). \tag{7.9}$$

A slight change in the argument of Section 6 leads in the same way to

$$p_{m^*}(t) = F_m{}^*(t) + O(e^{-mM}). \tag{7.10}$$

Lemma 7.3, along with (7.1) gives the required result.

(ii) Consider the $9l$ locii $\theta_j = \{t : \text{Re } Q(t) = jM/10l\}, j = 1,..., 9l$, which are closed, nested, surround S, and are contained in R'. Let the minimum distance between the adjacent locii be ξ. Then, since $D(t)$ is of degree $\leqslant 8l$, there must be at least one value j_0 of j for which all the zeros of D are at a distance of at least $\frac{1}{2}\xi$ from θ_{j_0}. For $t \in \theta_{j_0}$ we have, from (7.1), (7.9), and (7.10), that

$$\begin{aligned} |p_\nu(t)| &\leqslant \text{const}(\tfrac{1}{2}\xi)^{-8l}\, e^{(9/10)mM} \\ &\leqslant \text{const } e^{(9/10)mM} \end{aligned}$$

for sufficiently large m. The constant may be chosen to be independent of ν and t. The result now follows from the maxium modulus principle.

We are now in a position to prove the main result of the paper.

THEOREM 7.5. *Provided that Condition* 6.3 *is satisfied and that* S *consists of* l *components, the sequence of* $[N/N]$ *Frobenius Padé approximants to* $f(t)$ *converges in capacity to* $f(t)$ *as* $N \to \infty$, *in any closed, bounded domain* $R \subset S'$.

Proof. The result for an arbitrary N will follow if it is true for the basic integers ν, and for a closed, bounded simply connected domain S'. For this, using Lemmas 3.1 and 7.4, we have that

$$\begin{aligned} |f(t) - [\nu/\nu]| &\leqslant |p_\nu(t)|^{-1} r^{-1} \int_S |dt'|\, |\sigma(t')|\, |p_\nu(t')| \\ &\leqslant \text{const } r^{-1} d e^{-(1/10)mM} \delta_0^{-1}(\mu)\, I \end{aligned} \tag{7.11}$$

where

$$r = \inf_{\substack{t \in R \\ t' \in S}} |t - t'| \text{ and } I = \int_S |dt \, \sigma(t)|.$$

It is now obvious that for any $\mu, \epsilon > 0$, the right side of (7.10), by increasing m, i.e., ν, may be made $< \epsilon$, which proves the assertion. ∎

Appendix 1

The aim of this appendix is to find the equivalence class of a divisor $\alpha = \alpha_1, \ldots, \alpha_{l-1}$, that is, all divisors $\beta = \beta_1 \cdots \beta_{l-1}$ such that $\beta \sim \alpha$. For terminology and a description of the theorems used the reader is referred to Siegel [8].

We shall call two points γ_1, γ_2 on $\mathscr{R}$ a pair if $\gamma_1 = h(\gamma_2)$. They correspond to the same point in the complex plane but are on different sheets. We suppose that α contains m pairs and $l - 1 - 2m$ other unpaired points, with $m = 0, 1, \ldots$.

Now any differential of the first kind dw may be written as

$$dw = \pi(t) \, X^{-1/2}(t) \, dt$$

with π a degree $(l - 2)$ polynomial. If $\alpha \mid dw$, then π must have zeros in the complex plane at the unpaired points and also at the points corresponding to pairs, $m + (l - 1 - 2m) = l - 1 - m$ in all. Thus the dimension h of the vector space of first kind differentials such that $\alpha \mid dw$ is $h = l - 1 - (l - 1 - m) = m$. Suppose that a is the dimension of the vector space of functions of meromorphic on $\mathscr{R}$ such that $\alpha^{-1} \mid f$. The Riemann Roch theorem shows that

$$\begin{aligned} a &= h + l - 1 - (l - 1) + 1 \\ &= m + 1. \end{aligned}$$

A function f for which $\alpha^{-1} \mid f$ is

$$f = Q_m(t)/R_m(t)$$

where $R_m(t)$ is a degree m polynomial which has its zeros at the paired points of α, and $Q_m(t)$ is any degree m polynomial. The dimension of the vector space of such functions is $m + 1$, and since $a = m + 1$, it follows that every function for which $\alpha^{-1} \mid f$ has this form.

It follows that $\beta \sim \alpha$, if and only if $\beta = \alpha_1 \cdots \alpha_{l-1-2m} \beta_1 h(\beta_1) \cdots \beta_m h(\beta_m)$, with $\beta_1, \ldots, \beta_m$ arbitrary.

Appendix 2

In this appendix, we demonstrate that Condition 6.3(iv) follows from Condition 6.3(ii), (iii). Given a bounded function $\rho(t)$ satisfying, with $\rho = \sigma^{-1}$, (ii) and (iii) of Condition 6.3, we demonstrate that it is possible to find a polynomial $\rho_m(t)$ of degree m, such that, in the case when S consists of l separate arcs,

$$\sup_{t \in S} | \rho(t) - \rho_m(t)| < \text{const}(\ln m)^{-1-\lambda}. \tag{A2.1}$$

The proof uses the orthogonal polynomials of Section 5 for weight $\sigma = 1$. We shall use the notation of Section 6. Since $\chi_m(t) = 1$, we have

$$F(t) = e^{m\phi(t)} g(t).$$

There is a sequence of integers, which we call Σ, for which μ_m^{-1} is uniformly bounded.

The function $K_m(t, t')$ is constructed as in (6.2)

$$K_m(t, t') = q_m(t)\, q_{m+1}(t') - q_{m+1}(t)\, q_m(t'). \tag{A2.2}$$

Define the degree m polynomial $Q_m(t)$ by

$$Q_m(t) = \mu_m^{-1} \int_S dt'\, X_+^{-(1/2)}(t')\, K_m(t, t')(t' - t)^{-1} \rho(t') \tag{A2.3}$$

It follows as in Lemma 6.1 that

$$Q_m(t) - \rho(t) = \mu_m^{-1} \int_S dt'\, X_+^{-(1/2)}(t')\, K_m(t, t')(t' - t)^{-1} (\rho(t') - \rho(t)). \tag{A2.4}$$

To study this integral, we follow the method of Widom [17, Sects. 8, 11], Let us consider the contribution to (A2.4) from S_1, one component of S, which we shall assume ends at -1, 1 (d_1, d_2). We shall also assume $t \in S_1$. The change of variable

$$t = \tfrac{1}{2}(s + s^{-1}) = \psi(s)$$

may be taken to map the exterior of S_1 into the exterior of an analytic Jordan curve Γ_1 in the s-plane. The curve Γ_1 passes through the points 1, -1. Each arc of Γ_1 joining 1, -1 corresponding to s_1, in such a way that points s, $s^{-1} \in \Gamma_1$ correspond to the same point of s_1.

Equation (A2.4) may now be written

$$\begin{aligned} &Q_m(t) - \rho(t) \\ &\quad = -2\mu_m^{-1} \int_{\Gamma_1} ds'\, Y(s')\, K_m(t, \psi(s'))((s' - s)(s' - s^{-1}))^{-1} (\rho(\psi(s')) - \rho(t)) \end{aligned}$$

where

$$Y(s) = \sum_{i=3}^{2l} (\psi(s) - d_i)^{-(1/2)}.$$

Precisely as in Lemma 6.8, we write K_m as the sum of four terms, one of which is, with $t, t' \in S_1$,

$$\begin{aligned} &F_+(t)\, F_+'(t') - F_+'(t)\, F_-(t') \\ &\quad = e^{m(\phi_+(t) + \phi_+(t'))}(g_+(t)\, g_+'(t')\, e^{\phi_+(t')} - g_-'(t)\, g_+(t')\, e^{\phi_+(t)}) \\ &\quad = K \qquad \text{(say).} \end{aligned} \tag{A2.5}$$

If we substitute $t' = \psi(s')$, the factor in brackets in (A2.5) becomes an analytic function of s' in a neighborhood of Γ_1, which vanishes when $s' = s$. We may therefore write

$$K = (s' - s)\, e^{m(\phi_+(t) + \phi_+(t'))} \mathscr{F}(s', s)$$

where $\mathscr{F}(s', s)$ is uniformly bounded. For later use we note that

$$\begin{aligned} &Y(s^{-1})\, \mathscr{F}(s^{-1}, s) \\ &\quad = Y(s^{-1})(s^{-1} - s)^{-1}\, e^{-m(\phi_+(t) + \phi_-(t))}(F_+(t)\, F_-'(t) - F_+'(t)\, F_-(t)) \\ &\quad = -\tfrac{1}{2} X_+^{-(1/2)}(t)(F_+(t)\, F_-'(t) - F_+'(t)\, F_-(t))\, e^{-2m\phi(1)} \end{aligned}$$

where we have assumed that $\phi(t)$ is continuous as we pass round S_1 except at -1. Using Lemma 6.2 we find

$$Y(s^{-1})\, \mathscr{F}(s^{-1}, s) = -(1/2\pi i)\, \mu_m e^{2m\phi(1)}.$$

We now follow the procedure of Widom. If $2\pi\omega i$ is the change in $\phi(t)$ as we pass round S_1, we introduce a new variable z by

$$z = \exp[(1/\omega)(\phi(\psi(s)) - \phi(1))].$$

This is a conformal transformation of a neighborhood of Γ_1 into a neighborhood of the unit circle, which has an inverse transformation $s = \theta(z)$. Suppose that

$$m\omega^{-1} = n + \eta$$

where n is an integer and $|\eta| < 1$. Then we have that the contribution from K to (A2.4) is

$$\begin{aligned} I_m = -2\mu_m^{-1} e^{2m\phi(1)} \int_{|z'|=1} & dz'\, (d\theta/dz')\, Y(\theta(z'))\, z^n z'^n z^\eta z'^\eta \\ & \times \mathscr{F}(\theta(z'), s)(\theta(z') - s^{-1})^{-1}\, (\rho(\psi(\theta(z'))) - \rho(t)). \end{aligned}$$

We note that

$$z^{-1} = \exp[(1/\omega)(\phi(\psi(s^{-1})) - \phi(1))]$$

and we may write

$$\theta(z') - s^{-1} = (z' - z^{-1})\, \mathscr{G}(z', z)$$

where $\mathscr{G}(z', z)$ is analytic and nonzero near the unit circle. We have

$$\mathscr{G}(z^{-1}, z) = (d\theta/dz')|_{z=z^{-1}} .$$

The result is that we may write I_m as

$$\begin{aligned} I_m &= \int_{|z'|} dz'\, \mathscr{H}(z', z)(z' - z^{-1})^{-1} z^n z'^n(\rho(\psi(\theta(z'))) - \rho(t)) \\ &= \int_{|z'|=1} dz'\, (\mathscr{H}(z', z) - \mathscr{H}(z^{-1}, z))(z' - z^{-1})^{-1} z^n z'^n(\rho(\psi(\theta(z'))) - \rho(t)) \\ &\quad + \mathscr{H}(z^{-1}, z) \int_{|z'|=1} dz'\, z^n z'^n (z' - z^{-1})^{-1} (\rho(\psi(\theta(z'))) - \rho(t)). \end{aligned} \tag{A2.6}$$

Here, $\mathscr{H}(z', z)$ is analytic, with uniformly bounded derivative for z' near $|z'| = 1$, and

$$\mathscr{H}(z^{-1}, z) = (\pi i)^{-1}.$$

The assumed smoothness of $\rho(t)$ implies that the first term in (A2.6) is the Fourier coefficient of a function with modulus of continuity $\omega(\delta) <$ const $(\ln \delta)^{-1-\lambda}$, so that [18] this term is bounded by const$(\ln m)^{-1-\lambda}$. The contribution from the remainder of S has a similar bound. Widom shows that the second term may be written as $R_m(z^{-1})$, where

$$R_m(z) = \sum_{k=m+1}^{\infty} z^{-k} a_k ,$$

$$a_k = \frac{1}{2\pi i} \int_{|z'|=2} dz'\, z'^{k+1} A(z'),$$

with

$$A(z) = \frac{1}{\pi i} \int_{|z'|=1} dz'\, \frac{\rho(\psi(\theta(z')))}{z' - z} .$$

Thus $R_m(z)$ is the remainder after m terms of the function $A(z)$, analytic in $|z| > 1$, with modulus of continuity $<$ const$(\ln \delta)^{-1-\lambda}$ on $|z| = 1$. It follows immediately [19] that R_m is uniformly $O((\ln m)^{-\lambda})$, but this is not adequate for our requirements.

To construct the approximating polynomial of degree m, $P_m(t)$, we take a linear combination of polynomials $Q_k(t)$, $k \in \Sigma$, $k \leqslant m$, If $[m/2]$ is the greatest integer not greater than $m/2$, let $k_1, \ldots, k_\nu$ be those members of Σ satisfying

$$[m/2] \leqslant k_1 < k_2 < \cdots < m$$

so that, from Lemma 4.9, $k_{i+1} - k_i \leq l$. Let us define λ_i by

$$\lambda_i = (k_{i+1} - k_i)/(m - [m/2]) \qquad i = 2, \ldots, \nu,$$

where $k_{\nu+1}$ is taken to be m. We also set

$$\lambda_1 = 1 - \sum_{i=2}^{\nu} \lambda_i = \frac{k_2 - [m/2]}{m - [m/2]}.$$

Thus

$$0 \leq \lambda_i \leq 2l/m, \qquad i = 1, \ldots, \nu.$$

We set

$$P_m(t) = \sum_{i=1}^{\nu} \lambda_i Q_{k_i}(t).$$

From that part of K_m discussed above, it follows that

$$P_m(t) = \rho(t) + \sum_{i=1}^{\nu} \lambda_i R_{k_i}(z^{-1}) + \sum_{i=1}^{\nu} \lambda_i b_i + \text{other terms} \qquad \text{(A2.7)}$$

where

$$|b_i| < \text{const}(\ln k_i)^{-1-\lambda}$$

so that

$$\left| \sum_{i=1}^{\nu} \lambda_i b_i \right| < \text{const}(\ln m)^{-1-\lambda}.$$

The first term of (A2.7) is little different from the de la Vallée Poussin method of summing the Fourier series for $A(z^{-1})$. Indeed, we may write, using the definition of $D_{2m}(f, x)$ given by Feinerman and Newman [20],

$$\sum_{i=1}^{\nu} \lambda_i R_{k_i}(z^{-1}) = \{A(z^{-1}) - D_{2m}(A, z^{-1})\} + \sum_{j=[m/2]}^{m} \mu_j a_j z^{-j} \qquad \text{(A2.8)}$$

where

$$|\mu_j| < 2l/m.$$

Since a_j is the Fourier coefficient of a function with modulus of continuity $<\text{const}(\ln \delta)^{-1-\lambda}$, we have [18]

$$|a_j| < \text{const}(\ln j)^{-1-\lambda}$$

and the second term of (A2.8) satisfies

$$\left| \sum_{j=[m/2]}^{m} \mu_j a_j z^{-j} \right| < \text{const}(\ln m)^{-1-\lambda}.$$

Theorem 1 of Feinerman and Newman [20] gives the same bound on the first term of (A2.8).

The proof is completed by observing that the remaining parts of K_m may be treated in the same way.

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